

Complex Projective Structures

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1 Introduction

In this chapter we discuss the theory of complex projective structures on compact surfaces and its connections with Teichmüller theory, 2- and 3-dimensional hyperbolic geometry, and representations of surface groups into $\mathrm{PSL}_2(\mathbb{C})$. Roughly speaking, a complex projective structure is a type of 2-dimensional geometry in which Möbius transformations play the role of geometric congruences (this is made precise below). Such structures are abundant—hyperbolic, spherical, and Euclidean metrics on surfaces all provide examples of projective structures, since each of these constant-curvature 2-dimensional geometries has a model in which its isometries are Möbius maps. However, these examples are not representative of the general situation, since most projective structures are not induced by locally homogeneous Riemannian metrics.

Developing a more accurate picture of a general projective structure is the goal of the first half of the chapter (§§2–4). After some definitions and preliminary discussion (in §2), we present the complex-analytic theory of projective structures in §3. This theory has its roots in the study of automorphic functions and differential equations by Klein [67, Part 1], Poincaré [95], Riemann [100], and others in the late nineteenth century (see [48] [47, §1] for further historical discussion and references), while its more recent history is closely linked to developments in Teichmüller theory and deformations of Fuchsian and Kleinian groups (e.g. [28] [45] [43] [47] [51] [74] [75] [76] [77]).

In this analytic approach, a projective structure is represented by a holomorphic quadratic differential on a Riemann surface, which is extracted from the geometric data using a Möbius-invariant differential operator, the *Schwarzian derivative*. The inverse of this construction describes every projective structure in terms of holomorphic solutions to a linear ordinary differential equation (the *Schwarzian equation*). In this way, many properties of projective structures and their moduli can be established using tools from complex function theory. However, in spite of the success of these techniques, the analytic theory is somewhat detached from the underlying geometry. In particular, the analytic parameterization of projective structures does not involve an explicit geometric construction, such as one has in the description of hyperbolic surfaces by gluing polygons.

In §4 we describe a more direct and geometric construction of complex projective structures using *grafting*, a gluing operation on surfaces which is also suggested by the work of the nineteenth-century geometers (e.g. [68]), but whose significance in complex projective geometry has only recently been fully appreciated. Grafting was used by Maskit [83], Hejhal [47], and Sullivan-Thurston [109] to construct certain deformations of Fuchsian groups, and in later work of Thurston (unpublished, see [64]) it was generalized to give a universal construction of complex projective surfaces starting from basic hyperbolic and Euclidean pieces.

This construction provides another coordinate system for the moduli space of projective structures, and it reveals an important connection between these structures and convex geometry in 3-dimensional hyperbolic space. However, the explicit geometric nature of complex projective grafting comes at the price of a more complicated parameter space, namely, the piecewise linear manifold of *measured geodesic laminations* on hyperbolic surfaces. In particular, the lack of a differentiable structure in this coordinate system complicates the study of variations of complex projective structures, though there has been some progress in this direction using a weak notion of differentiability due to Thurston [115] and Bonahon [10].

After developing the analytic and geometric coordinates for the moduli space of projective structures, the second half of the chapter is divided into two major topics: In §5, we describe the relation between projective structures and the $\mathrm{PSL}_2(\mathbb{C})$ -representations of surface groups, their deformations, and associated problems in hyperbolic geometry and Kleinian groups. The key to these connections is the *holonomy representation* of a projective structure, which records the topological obstruction to analytically continuing its local coordinate charts over the entire surface. After constructing a parameter space for such representations and the holonomy map for projective structures, we survey various developments that center around two basic questions:

- Given a projective structure, described in either analytic or geometric terms, what can be said about its holonomy representation?
- Given a $\mathrm{PSL}_2(\mathbb{C})$ -representation of a surface group, what projective structures have this as their holonomy representation, if any?

We discuss partial answers to these general questions, along with much more detailed information about certain classes of holonomy representations (e.g. Fuchsian groups).

Finally, in §6 we take up the question of relating the analytic and geometric coordinate systems for the space of projective structures, or equivalently, studying the interaction between the Schwarzian derivative and complex projective grafting. We describe asymptotic results that relate compactifications of the analytic and geometric parameter spaces using the geometry of measured foliations on Riemann surfaces. Here a key tool is the theory of harmonic maps between Riemann surfaces and from Riemann surfaces to \mathbb{R} -trees, and the observation that two geometrically natural constructions in complex projective geometry (the *collapsing* and *co-collapsing* maps) are closely approximated by harmonic maps. We close with some remarks concerning infinitesimal compatibility between the geometric and analytic coordinate systems, once again using the limited kind of differential calculus that applies to the grafting parameter space.

Scope and approach. Although this chapter covers a range of topics in complex projective geometry, is not intended to be a comprehensive guide to the

subject. Rather, we have selected several important aspects of the theory (the Schwarzian derivative, grafting, and holonomy) and concentrated on describing their interrelationships while providing references for further reading and exploration. As a result, some major areas of research in complex projective structures are not mentioned at all (circle packings [70] [69], the algebraic-geometric aspects of the theory [36, §11], and generalizations to punctured or open Riemann surfaces [75] [81], to name a few) and others are only discussed in brief.

We have also included some detail on the basic analytic and geometric constructions in an attempt to make this chapter a more useful “invitation” to the theory. However, where we discuss more advanced topics and results of recent research, it has been necessary to refer to many concepts and results that are not thoroughly developed here.

Finally, while we have attempted to provide thorough and accurate references to the literature, the subject of complex projective structures is broad enough (and connected to so many other areas of research) that we do not expect these references to cover every relevant source of additional information. We hope that the references included below are useful, and regret any inadvertent omissions.

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2 Basic definitions

Projective structures. Let S be an oriented surface. A *complex projective structure* Z on S is a maximal atlas of charts mapping open sets in S into \mathbb{CP}^1 such that the transition functions are restrictions of Möbius transformations. For brevity we also call these *projective structures* or \mathbb{CP}^1 -*structures*.

We often treat a projective structure Z on S as a surface in its own right—a *complex projective surface*. Differentiably, Z is the same as S , but Z has the additional data of a restricted atlas of projective charts.

Two projective structures Z_1 and Z_2 on S are *isomorphic* if there is an orientation-preserving diffeomorphism $\iota : Z_1 \rightarrow Z_2$ that pulls back the projective charts of Z_2 to projective charts of Z_1 , and *marked isomorphic* if furthermore ι is homotopic to the identity.

Our main object of study is the space $\mathcal{P}(S)$ of marked isomorphism classes of projective structures on a compact surface S . Thus far, we have only defined

$\mathcal{P}(S)$ as a set, but later we will equip it with the structure of a complex manifold.

Non-hyperbolic cases. Projective structures on compact surfaces are most interesting when S has genus $g \geq 2$: The sphere has a unique projective structure (by $S^2 \simeq \mathbb{CP}^1$) up to isotopy, while a projective structure on a torus is always induced by an affine structure [43, §9, pp. 189-191]. We therefore make the assumption that S has genus $g \geq 2$ unless stated otherwise.

First examples. The projective structure of \mathbb{CP}^1 itself (using the identity for chart maps) also gives a natural projective structure on any open set $U \subset \mathbb{CP}^1$. If U is preserved by a group Γ of Möbius transformations acting freely and properly discontinuously, then the quotient surface $X = U/\Gamma$ has a natural projective structure in which the charts are local inverses of the covering $U \rightarrow X$.

In particular any Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ gives rise to a projective structure on the quotient surface \mathbb{H}/Γ and a Kleinian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ gives a projective structure on the quotient of its domain of discontinuity $\Omega(\Gamma)/\Gamma$. Rephrasing the latter example, the ideal boundary of a hyperbolic 3-manifold has a natural projective structure.

Locally Möbius maps. A map $f : Z \rightarrow W$ between complex projective surfaces is *locally Möbius* if for every sufficiently small open set $U \subset Z$, the restriction $f|_U$ is a Möbius transformation with respect to projective coordinates on U and $f(U)$. Examples of such maps include isomorphisms and covering maps of projective surfaces (where the cover is given the pullback projective structure) and inclusions of open subsets of surfaces.

Developing maps. A projective structure Z on a surface S lifts to a projective structure \tilde{Z} on the universal cover \tilde{S} . A *developing map* for Z is an immersion $f : \tilde{S} \rightarrow \mathbb{CP}^1$ such that the restriction of f to any sufficiently small open set in \tilde{S} is a projective chart for \tilde{Z} . Such a map is also called a *geometric realization* of Z (e.g. [45, §6]) or a *fundamental membrane* [47].

Developing maps always exist, and are essentially unique—two developing maps for a given structure differ by post-composition with a Möbius transformation. Concretely, a developing map can be constructed by analytic continuation starting from any basepoint $z_0 \in \tilde{Z}$ and any chart defined on a neighborhood U of z_0 . Another chart $V \rightarrow \mathbb{CP}^1$ that overlaps U can be adjusted by a Möbius transformation so as to agree on the overlap, gluing to give a map $(U \cup V) \rightarrow \mathbb{CP}^1$. Continuing in this way one defines a map on successively larger subsets of \tilde{Z} , and the limit is a developing map $\tilde{Z} \rightarrow \mathbb{CP}^1$. The simple connectivity of \tilde{Z} is essential here, as nontrivial homotopy classes of loops in the surface create obstructions to unique analytic continuation of a projective chart.

For a fixed projective structure, we will speak of *the* developing map when the particular choice is unimportant or implied.

Holonomy representation. The developing map $f : \tilde{S} \rightarrow \mathbb{CP}^1$ of a projective structure Z on S has an equivariance property with respect to the action of $\pi_1(S)$ on \tilde{S} : For any $\gamma \in \pi_1(S)$, the composition $f \circ \gamma$ is another developing map for Z . Thus there exists $A_\gamma \in \mathrm{PSL}_2(\mathbb{C})$ such that

$$f \circ \gamma = A_\gamma \circ f \quad (2.1)$$

The map $\gamma \mapsto A_\gamma$ is a homomorphism $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$, the *holonomy representation* (or *monodromy representation*) of the projective structure.

Development-holonomy pairs. The developing map and holonomy representation form the *development-holonomy pair* (f, ρ) associated to the projective structure Z .

This pair determines Z uniquely, since restriction of f determines a covering of S by projective charts. Post-composition of the developing map with $A \in \mathrm{PSL}_2(\mathbb{C})$ conjugates ρ , and therefore the pair (f, ρ) is uniquely determined by Z up to the action of $\mathrm{PSL}_2(\mathbb{C})$ by

$$(f, \rho) \mapsto (A \circ f, \rho^A) \text{ where } \rho^A(\gamma) = A\rho(\gamma)A^{-1}$$

Conversely, any pair (f, ρ) consisting of an immersion $f : \tilde{S} \rightarrow \mathbb{CP}^1$ and a homomorphism $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ that satisfy (2.1) defines a projective structure on S in which lifting $U \subset S$ to \tilde{S} and applying f gives a projective chart (for all sufficiently small open sets U).

Thus we have an alternate definition of $\mathcal{P}(S)$ as the quotient of the set of development-holonomy pairs by the $\mathrm{PSL}_2(\mathbb{C})$ action and by precomposition of developing maps with orientation-preserving diffeomorphisms of S homotopic to the identity. We give the set of pairs of maps (f, ρ) the compact-open topology, and $\mathcal{P}(S)$ inherits a quotient topology. We will later see that $\mathcal{P}(S)$ is homeomorphic to \mathbb{R}^{12g-12} .

Relation to (G, X) -structures. There is a very general notion of a geometric structure defined by a Lie group G acting by diffeomorphisms on a manifold X . A (G, X) -*structure* on a manifold M is an atlas of charts mapping open subsets of M into X such that the transition maps are restrictions of elements of G .

In this language, complex projective structures are $(\mathrm{PSL}_2(\mathbb{C}), \mathbb{CP}^1)$ -structures. Some of the properties of projective structures we develop, such as developing maps, holonomy representations, deformation spaces, etc., can be applied in the more general setting of (G, X) -structures. See [40] for a survey of (G, X) -structures and analysis of several low-dimensional examples.

Circles. Because Möbius transformations map circles to circles, there is a natural notion of a circle on a surface with a projective structure Z : A smooth

embedded curve $\alpha \subset Z$ is a *circular arc* if the projective charts map (subsets of) α to circular arcs in \mathbb{CP}^1 . Equivalently, the embedded curve α is a circular arc if the developing map sends any connected component of the preimage of α in \tilde{Z} to a circular arc in \mathbb{CP}^1 . A closed circular arc on Z is a *circle*.

Small circles are ubiquitous in any projective structure: For any $z \in Z$ there is a projective chart mapping a contractible neighborhood of z to an open set $V \subset \mathbb{CP}^1$. The preimage of any circle contained in V is a homotopically trivial circle for the projective structure Z . Circles that bound disks on a projective surface have an important role in Thurston's projective grafting construction (see §4.1).

Circles on a projective surface can also be homotopically nontrivial. For example, any simple closed geodesic on a hyperbolic surface is a circle, because its lifts to \mathbb{H} are half-circles or vertical lines in the upper half-plane. The analysis of circles on more general projective surfaces would be a natural starting point for the development of synthetic complex projective geometry; Wright's study of circle chains and Schottky-type dynamics in the Maskit slice of punctured tori is an example of work in this direction [120].

Forgetful map. Since Möbius transformations are holomorphic, a projective structure $Z \in \mathcal{P}(S)$ also determines a complex structure, making S into a compact Riemann surface. In this way, marked isomorphism of projective structures corresponds to marked isomorphism of Riemann surfaces, and so there is a natural (and continuous) *forgetful map*

$$\pi : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$$

where $\mathcal{T}(S)$ is the Teichmüller space of marked isomorphism classes of complex structures on S . (See e.g. [79], [56], [52] for background on Teichmüller spaces.) As a matter of terminology, if Z is a projective structure with $\pi(Z) = X$, we say Z is a projective structure *on the Riemann surface* X .

The forgetful map is surjective: By the uniformization theorem, every complex structure $X \in \mathcal{T}(S)$ arises as the quotient of \mathbb{H} by a Fuchsian group Γ_X , and the natural projective structure on \mathbb{H}/Γ_X is a preimage of X by π . We call this the *standard Fuchsian structure* on X . The standard Fuchsian structures determine a continuous section

$$\sigma_0 : \mathcal{T}(S) \rightarrow \mathcal{P}(S).$$

One might expect the fibers of π to be large, since isomorphism of projective structures is a much stronger condition than isomorphism of complex structures. Our next task is to describe the fibers explicitly.

3 The Schwarzian Parameterization

3.1 The Schwarzian derivative

Let $\Omega \subset \mathbb{C}$ be a connected open set. The *Schwarzian derivative* of a locally injective holomorphic map $f : \Omega \rightarrow \mathbb{CP}^1$ is the holomorphic quadratic differential

$$S(f) = \left[\left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right] dz^2.$$

Two key properties make the Schwarzian derivative useful in the theory of projective structures:

- (1) *Cocycle property.* If f and g are locally injective holomorphic maps such that the composition $f \circ g$ is defined, then

$$S(f \circ g) = g^* S(f) + S(g)$$

- (2) *Möbius invariance.* For any $A \in \mathrm{PSL}_2(\mathbb{C})$, we have

$$S(A) \equiv 0,$$

and conversely, if $S(f) \equiv 0$, then f is the restriction of a Möbius transformation.

Note that the pullback $g^* S(f)$ uses the definition of the Schwarzian as a quadratic differential. In classical complex analysis, the Schwarzian was regarded as a complex-valued function, with $g^* S(f)$ replaced by $g'(z)^2 S(f)(g(z))$.

An elementary consequence of these properties is that the map f is almost determined by its Schwarzian derivative; if $S(f) = S(g)$, then the locally defined map $f \circ g^{-1}$ satisfies $S(f \circ g^{-1}) \equiv 0$, and so we have $f = A \circ g$ for some $A \in \mathrm{PSL}_2(\mathbb{C})$.

Further discussion of the Schwarzian derivative can be found in e.g. [79, Ch. 2] [52, §6.3].

Osculation. Intuitively, the Schwarzian derivative measures the failure of a holomorphic map to be the restriction of a Möbius transformation. Thurston made this intuition precise as follows (see [116, §2], [2, §2.1]): For each $z \in \Omega$, there is a unique Möbius transformation that has the same 2-jet as f at z , called the *osculating Möbius transformation* $\mathrm{osc}_z f$.

The *osculation map* $G : \Omega \rightarrow \mathrm{PSL}_2(\mathbb{C})$ given by $G(z) = \mathrm{osc}_z f$ is holomorphic, and its Darboux derivative (see [105]) is the holomorphic $\mathfrak{sl}_2(\mathbb{C})$ -valued 1-form

$$\omega(z) = G^{-1}(z) dG(z).$$

An explicit computation shows that ω only depends on f through its Schwarzian derivative; if $S(f) = \phi(z)dz^2$, then

$$\omega(z) = -\frac{1}{2}\phi(z) \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} dz.$$

3.2 Schwarzian parameterization of a fiber

Fibers over Teichmüller space For any marked complex structure $X \in \mathcal{T}(S)$, let $P(X) = \pi^{-1}(X) \subset \mathcal{P}(S)$ denote the set of marked complex projective structures with underlying complex structure X . The Schwarzian derivative can be used to parameterize the fiber $P(X)$ as follows:

Fix a conformal identification $\tilde{X} \simeq \mathbb{H}$, whereby $\pi_1(S)$ acts on \mathbb{H} as a Fuchsian group. Abusing notation, we use the same symbol for $\gamma \in \pi_1(S)$ and for its action on \mathbb{H} by a real Möbius transformation.

Given $Z \in P(X)$, we regard the developing map as a meromorphic function f on \mathbb{H} . The Schwarzian derivative $\tilde{\phi} = S(f)$ is therefore a holomorphic quadratic differential on \mathbb{H} . Combining the equivariance property (2.1) of f and the properties of the Schwarzian derivative, we find

$$\tilde{\phi} = S(A_\gamma \circ f) = S(f \circ \gamma) = \gamma^* \tilde{\phi},$$

Thus we have $\tilde{\phi} = \gamma^* \tilde{\phi}$ for all $\gamma \in \pi_1(S)$, and $\tilde{\phi}$ descends to a holomorphic quadratic differential ϕ on X . We call ϕ the *Schwarzian of the projective structure* Z .

Let $Q(X)$ denote the vector space of holomorphic quadratic differentials on the marked Riemann surface $X \in \mathcal{T}(S)$. By the Riemann-Roch theorem, we have $Q(X) \simeq \mathbb{C}^{3g-3}$ (see [62]). The Schwarzian defines a map $P(X) \rightarrow Q(X)$. We will now show that this map is bijective by constructing its inverse.

Inverting the Schwarzian. Let $\phi(z)$ be a holomorphic function defined on a contractible open set $\Omega \subset \mathbb{C}$. Then the linear ODE (the *Schwarzian equation*)

$$u''(z) + \frac{1}{2}\phi(z)u(z) = 0 \tag{3.1}$$

has a two-dimensional vector space V of holomorphic solutions on Ω . Let $u_1(z)$ and $u_2(z)$ be a basis of solutions. The Wronskian $W(z)$ of u_1 and u_2 satisfies $W'(z) = 0$, so it is a nonzero constant function, and u_1 and u_2 cannot vanish simultaneously.

This ODE construction inverts the Schwarzian derivative in the sense that the meromorphic function $f(z) = u_1(z)/u_2(z)$ satisfies $S(f) = \phi(z) dz^2$ (see [91]). Note that changing the basis for V will alter f by composition with a

Möbius transformation (and leave $S(f)$ unchanged). Furthermore, since

$$f'(z) = \frac{u_1'(z)u_2(z) - u_1(z)u_2'(z)}{u_2(z)^2} = \frac{-W(z)}{u_2(z)^2},$$

it follows that the holomorphic map $f : \Omega \rightarrow \mathbb{CP}^1$ is locally injective except possibly on $\{u_2(z) = 0\} = f^{-1}(\infty)$. Applying similar considerations to $1/f(z)$, we find that f is locally injective away from $\{u_1(z) = 0\}$, and thus everywhere.

The existence of a holomorphic map with a given Schwarzian derivative can also be understood in terms of maps to the Lie group $\mathrm{PSL}_2(\mathbb{C})$ and the definition of the Schwarzian in terms of osculation (described in §3.1). Here the quadratic differential ϕ is interpreted as a $\mathfrak{sl}_2(\mathbb{C})$ -valued 1-form, which satisfies the integrability condition $d\phi + \frac{1}{2}[\phi, \phi] = 0$ because there are no holomorphic 2-forms on a Riemann surface. The integrating map to $\mathrm{PSL}_2(\mathbb{C})$ is the osculation map of a holomorphic function f satisfying $S(f) = \phi$. See [2, §2.2.3, Cor 2.20] for details.

Parameterization of a fiber. Given a quadratic differential $\phi \in Q(X)$, lift to the universal cover $\tilde{X} \simeq \mathbb{H}$ to obtain $\tilde{\phi} = \tilde{\phi}(z) dz^2$. Applying the ODE construction to $\tilde{\phi}(z)$ yields a holomorphic immersion $f_\phi : \mathbb{H} \rightarrow \mathbb{CP}^1$.

For any $\gamma \in \pi_1(S)$ we have $S(f_\phi \circ \gamma) = \gamma^* \tilde{\phi} = \tilde{\phi} = S(f_\phi)$, and thus $f_\phi \circ \gamma = A_\gamma \circ f_\phi$ for some $A_\gamma \in \mathrm{PSL}_2(\mathbb{C})$. We set $\rho_\phi(\gamma) = A_\gamma$. Then (f_ϕ, ρ_ϕ) determine a development-holonomy pair, and thus a projective structure X_ϕ on S . Since f is holomorphic, we also have $\pi(X_\phi) = X$.

The map $Q(X) \rightarrow P(X)$ given by $\phi \mapsto X_\phi$ is inverse to the Schwarzian map $P(X) \rightarrow Q(X)$ because the ODE construction is inverse to the Schwarzian derivative. In particular, each fiber of $\pi : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$ is naturally parameterized by a complex vector space.

Affine naturality. The identification $Q(X) \simeq P(X)$ defined above depends on a choice of coordinates on the universal cover of X . Specifically, we computed the Schwarzian using the coordinate z of the upper half-plane.

A coordinate-independent statement is that the Schwarzian derivative is a measure of the *difference* between a pair of projective structures on X , which we can see as follows: Given $Z_1, Z_2 \in P(X)$, let U be a sufficiently small open set on S so that there are projective coordinate charts $z_i : U \rightarrow \mathbb{CP}^1$ of Z_i for $i = 1, 2$. We can assume that $\infty \notin z_i(U)$.

The quadratic differential $z_1^* S(z_2 \circ z_1^{-1})$ on U is holomorphic with respect to the Riemann surface structure X . Covering S by such sets, it follows from the cocycle property that these quadratic differentials agree on overlaps and define an element $\phi \in Q(X)$, which is the *Schwarzian of Z_2 relative to Z_1* . Abusing notation, we write $Z_2 - Z_1 = \phi$.

Thus $P(X)$ has a natural structure of an *affine space* modeled on the vector space $Q(X)$. The choice of a basepoint $Z_0 \in P(X)$ gives an isomorphism $P(X) \rightarrow Q(X)$, namely $Z \mapsto (Z - Z_0)$. See [51, §2] for details.

From this perspective, the previous identification $P(X) \rightarrow Q(X)$ using the Schwarzian of the developing map on \mathbb{H} is simply $Z \mapsto (Z - \sigma_0(X))$, that is, it is the Schwarzian relative to the standard Fuchsian structure. Complex-analytically, this is not the most natural way to choose a basepoint in each fiber, though this will be remedied below (§3.3).

The realization of $P(X)$ as an affine space modeled on a vector space of differential forms can also be understood in terms of Čech cochains on X with a fixed coboundary [45, §3], or in terms of connections on a principal $\mathrm{PSL}_2(\mathbb{C})$ -bundle of *projective frames* [2, §2.2] (and the related notions of the *graph* of a projective structure [40, §2] and of *\mathfrak{sl}_2 -opers* [35, §8.2]).

3.3 Schwarzian parameterization of $\mathcal{P}(S)$

Identification of bundles. There is a complex vector bundle $\mathcal{Q}(S) \rightarrow \mathcal{T}(S)$ over Teichmüller space whose total space consists of pairs (X, ϕ) , where $X \in \mathcal{T}(S)$ and $\phi \in Q(X)$. In Teichmüller theory, this bundle is identified with the holomorphic cotangent bundle of Teichmüller space (see e.g. [56] [52]). Since Teichmüller space is diffeomorphic to \mathbb{R}^{6g-6} , the bundle $\mathcal{Q}(S)$ is diffeomorphic to \mathbb{R}^{12g-12} .

Using a section $\sigma : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ to provide basepoints for the fibers, we can form a bijective *Schwarzian parameterization*

$$\begin{aligned} \mathcal{P}(S) &\longrightarrow \mathcal{Q}(S) \\ Z &\longmapsto (\pi(Z), Z - \sigma(\pi(Z))) \end{aligned}$$

which is compatible with the maps of these spaces to $\mathcal{T}(S)$. This correspondence identifies the zero section of $\mathcal{Q}(S)$ with the section σ of $\mathcal{P}(S)$. A different section σ will result in a parameterization that differs by a translation in each fiber.

Compatibility. The topology on $\mathcal{P}(S)$ defined using development-holonomy pairs is compatible with the topology of $\mathcal{Q}(S)$, in that the bijection induced by any continuous section $\sigma : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ is a homeomorphism. Continuity in one direction is elementary complex analysis—uniformly close holomorphic developing maps have uniformly close derivatives (on a smaller compact set), and therefore uniformly close Schwarzian derivatives, making $\mathcal{P}(S) \rightarrow \mathcal{Q}(S)$ continuous. On the other hand, continuity of $\mathcal{Q}(S) \rightarrow \mathcal{P}(S)$ follows from continuous dependence of solutions to the ODE (3.1) on its parameter ϕ .

Holomorphic structure. The bundle $\mathcal{Q}(S)$ is a complex manifold, and a holomorphic vector bundle over $\mathcal{T}(S)$. The Schwarzian parameterization given by a section $\sigma : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ transports these structures to $\mathcal{P}(S)$. However, two sections σ_1 and σ_2 induce the same complex structure on $\mathcal{P}(S)$ if and only if $(\sigma_1 - \sigma_2)$ is a holomorphic section of $\mathcal{Q}(S)$.

There is also a natural complex structure on $\mathcal{P}(S)$ that is defined without reference to its parameterization by $\mathcal{Q}(S)$: The tangent space $T_Z\mathcal{P}(S)$ can be identified with the cohomology group $H^1(Z, \mathcal{V}_{proj})$, where \mathcal{V}_{proj} is the sheaf of projective vector fields over Z , i.e. vector fields that in a local projective coordinate are restrictions of infinitesimal Möbius transformations. This cohomology group is complex vector space, which gives an integrable almost complex structure $J : T_Z\mathcal{P}(S) \rightarrow T_Z\mathcal{P}(S)$. (Compare the construction of [51, Prop. 1,2].)

Quasi-Fuchsian sections. Using deformations of Kleinian surface groups, we can construct a class of sections of $\mathcal{P}(S)$ that transport the complex structure of $\mathcal{Q}(S)$ to the natural complex structure on $\mathcal{P}(S)$. Given $X, Y \in \mathcal{T}(S)$, let $Q(X, Y)$ denote the quasi-Fuchsian group (equipped with an isomorphism $\pi_1(S) \simeq Q(X, Y)$) that simultaneously uniformizes X and Y (see e.g. [56, Ch. 6]). This means that $Q(X, Y)$ has domain of discontinuity $\Omega_+ \sqcup \Omega_-$ with marked quotient Riemann surfaces

$$\Omega_+/Q(X, Y) \simeq X \quad \Omega_-/Q(X, Y) \simeq \bar{Y}$$

where \bar{Y} is the complex conjugate Riemann surface of Y , which appears in the quotient because the induced orientation on the marked surface $\Omega_-/Q(X, Y)$ is opposite that of S .

As a quotient of a domain by a Kleinian group, the surface $\Omega_+/Q(X, Y)$ also has a natural projective structure, which we denote by $\Sigma_Y(X)$. By definition, the underlying Riemann surface of $\Sigma_Y(X)$ is X , so for any fixed $Y \in \mathcal{T}(S)$ this defines a *quasi-Fuchsian section*

$$\Sigma_Y : \mathcal{T}(S) \rightarrow \mathcal{P}(S).$$

These quasi-Fuchsian sections induce the natural complex structure on $\mathcal{P}(S)$. We sketch two ways to see this: First, Hubbard uses a cohomology computation to show that a section induces the canonical complex structure if and only if it can be represented by a *relative projective structure* on the universal curve over $\mathcal{T}(S)$ [51, Prop. 1,2]. The quasi-Fuchsian groups provide such a structure due to the analytic dependence of the solution of the Beltrami equation on its parameters [1], and the associated construction of the *Bers fiber space* [6].

Alternatively, one can show (as in the respective computations of Hubbard [51] and Earle [28]) that both the canonical complex structure on $\mathcal{P}(S)$ and the complex structure coming from a quasi-Fuchsian section make the holonomy map (discussed in §5) a local biholomorphism, and therefore they are holomorphically equivalent.

Norms. A norm on the vector space $Q(X)$ induces a natural measure of the “complexity” of a projective structure on X (relative to the standard Fuchsian

structure), or of the difference between two projective structures. There are several natural choices for such a norm.

The hyperbolic L^∞ norm $\|\phi\|_\infty$ is the supremum of the function $|\phi|/\rho^2$, where ρ^2 is the area element of the hyperbolic metric on X . Lifting ϕ to the universal cover and identifying $\tilde{X} \rightarrow \Delta$, we have

$$\|\phi\|_\infty = \|\tilde{\phi}\|_\infty = \frac{1}{4} \sup_{z \in \Delta} |\tilde{\phi}(z)|(1 - |z|^2)^2.$$

By Nehari's theorem, a holomorphic immersion $f : \Delta \rightarrow \mathbb{CP}^1$ satisfying $\|S(f)\|_\infty \leq \frac{1}{2}$ is injective, while any injective map satisfies $\|S(f)\|_\infty \leq \frac{3}{2}$ (see [91], also [96] [79]). More generally, the norm $\|S(f)\|_\infty$ gives a coarse estimate of the size of hyperbolic balls in Δ on which f is univalent [74, §3] [77, Lem. 5.1]. Thus, when applied to projective structures, the L^∞ norm reflects the geometry and valence of the developing map.

In Teichmüller theory, it is more common to use the L^1 norm $\|\phi\|_1$, which is the area of the surface X with respect to the singular Euclidean metric $|\phi|$. This norm is conformally natural, since it does not depend on the choice of a Riemannian metric on X . However, the intrinsic meaning of the L^1 norm of the Schwarzian derivative is less clear.

More generally, given any background Riemannian metric on X compatible with its conformal structure, there is an associated L^p norm on $Q(X)$. These norms, with $p \in (1, \infty)$ and especially $p = 2$, can be used to apply PDE estimates to the study of projective structures, as discussed in §6.4 below.

Note that while any two norms on the finite-dimensional vector space $Q(X)$ are bilipschitz equivalent, the bilipschitz constants between the L^∞ , L^1 , and hyperbolic L^p norms on $Q(X)$ diverge as $X \rightarrow \infty$ in Teichmüller space.

4 The Grafting Parameterization

4.1 Definition of grafting

Grafting is a geometric operation that can be used to build an arbitrary projective structure by gluing together simple pieces. We start by defining grafting in a restricted setting, and then work toward the general definition.

Grafting simple geodesics. Equip a Riemann surface $X \in \mathcal{T}(S)$ with its hyperbolic metric. Let γ be a simple closed hyperbolic geodesic on X . The basic grafting construction replaces γ with the cylinder $\gamma \times [0, t]$ to obtain a new surface $\text{gr}_{t\gamma}X$, the *grafting of X by $t\gamma$* , as shown in Figure 1. The natural metric on this surface is partially hyperbolic (on $X - \gamma$) and partially Euclidean (on the cylinder), and underlying this metric is a well-defined conformal structure on $\text{gr}_{t\gamma}X$.

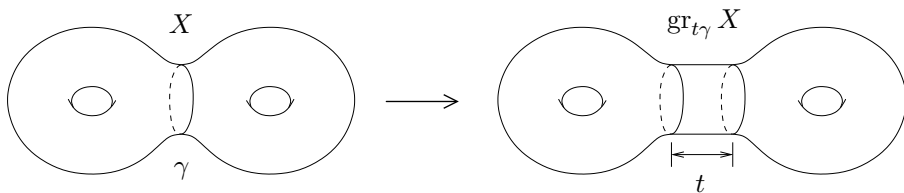


Figure 1. Grafting along a simple closed curve.

Let \mathcal{S} denote the set of free homotopy classes of homotopically nontrivial simple closed curves on S . Then \mathcal{S} is canonically identified with the set of simple closed geodesics for any hyperbolic structure on S , and we can regard grafting as a map

$$\text{gr} : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \rightarrow \mathcal{T}(S).$$

When it is important to distinguish this construction from the projective version defined below, we will call this *conformal grafting*, since the result is a conformal structure.

Projective grafting. The Riemann surface X has a standard Fuchsian projective structure in which the holonomy of a simple closed geodesic γ is conjugate to $z \mapsto e^\ell z$, where $\ell = \ell(\gamma, X)$ is the hyperbolic length of γ .

For any $t < 2\pi$, let \widetilde{A}_t denote a sector of angle t in the complex plane, with its vertex at 0. The quotient $A_t = \widetilde{A}_t / \langle z \mapsto e^\ell z \rangle$ is an annulus equipped with a projective structure, which as a Riemann surface is isomorphic to the Euclidean product $\gamma \times [0, t]$.

There is a natural projective structure on the grafted surface $\text{gr}_{t\gamma} X$ that is obtained by gluing the standard Fuchsian projective structure of X to A_t ; these structures are compatible due to the matching holonomy around the gluing curves. In the universal cover of X , this corresponds to inserting a copy of \widetilde{A}_t in place of each lift of γ (see Figure 2), applying Möbius transformations to \widetilde{A}_t and the complementary regions of γ in \widetilde{X} (which are bounded by circular arcs) so that they fit together. For sufficiently small t , this produces a Jordan domain in \mathbb{CP}^1 that is the image of the developing map, while for large t the developing image is all of \mathbb{CP}^1 . We denote the resulting projective structure by $\text{Gr}_{t\gamma} X$.

Applying a generic Möbius transformation to the sector \widetilde{A}_t will map it to a *t-lune*, the intersection of two round disks with interior angle t . Thus the projective structure $\text{Gr}_{t\gamma} X$ corresponds to a decomposition of its universal cover into *t-lunes* and regions bounded by circular arcs.

The restriction to small values of t in this construction is not necessary; for $t > 2\pi$ we simply interpret \widetilde{A}_t as a “sector” that wraps around the punctured

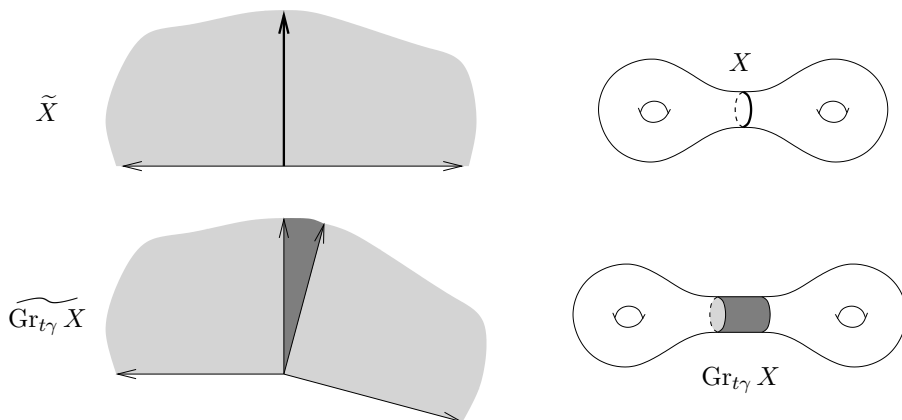


Figure 2. Projective grafting: Gluing a cylinder into the surface along a geodesic corresponds to inserting a sector or lune into each lift of the geodesic. Only one lift is shown here, but the gluing construction is repeated equivariantly in $\widetilde{\text{Gr}_{t\gamma} X}$.

plane \mathbb{C}^* some number of times. Alternatively, we could define A_t for $t \geq 2\pi$ by gluing n copies of $A_{t/n}$ end-to-end, for a sufficiently large $n \in \mathbb{N}$.

Therefore we have a *projective grafting map*,

$$\text{Gr} : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \rightarrow \mathcal{P}(S)$$

which is a lift of grafting through the forgetful map $\pi : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$, i.e. $\pi \circ \text{Gr} = \text{gr}$.

Variations on simple grafting. Grafting along a simple geodesic with weight $t = 2\pi$ was originally used by Maskit [83], Hejhal [47], and Sullivan-Thurston [109] to construct examples of exotic Fuchsian projective structures (discussed in §5.4 below). Grafting with weight 2π is special because it does not change the holonomy representation of the Fuchsian projective structure (see §5).

It is possible to extend this holonomy-preserving grafting operation to certain simple closed curves which are not geodesic, and to projective structures that are not standard Fuchsian (see [66, Ch. 7]); this generalization has been important to some applications in Kleinian groups and hyperbolic geometry (e.g. [13][11, §5]), and it will appear again in our description of quasi-Fuchsian projective structures (§5.5). However, our main focus in this chapter is a different extension of grafting, defined by Thurston, which leads to a geometric model for the entire moduli space $\mathcal{P}(S)$.

Extension to laminations. Projective grafting is compatible with the natural completion of $\mathbb{R}^+ \times \mathcal{S}$ to the space $\mathcal{ML}(S)$ of *measured laminations*. An element $\lambda \in \mathcal{ML}(S)$ is realized on a hyperbolic surface $X \in \mathcal{T}(S)$ as a foliation of a closed subset of X by complete, simple hyperbolic geodesics (some of which may be closed), equipped with a transverse measure of full support. A piecewise linear coordinate atlas for $\mathcal{ML}(S)$ is obtained by integrating transverse measures over closed curves, making $\mathcal{ML}(S)$ into a *PL*-manifold homeomorphic to \mathbb{R}^{6g-6} . See [113, Ch. 8-9] [18] [94, Ch. 3] for detailed discussion of measured laminations.

There is continuous extension $\text{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ of projective grafting, which is uniquely determined by the simple grafting construction because weighted simple curves are dense in $\mathcal{ML}(S)$. Similarly, there is an extension of the grafting map $\text{gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ defined by $\text{gr} = \pi \circ \text{Gr}$. These extensions were defined by Thurston [unpublished], and are discussed in detail in [64].

For a lamination $\lambda \in \mathcal{ML}(S)$ that is supported on a finite set of disjoint simple closed curves, i.e. $\lambda = \sum_{i=1}^n t_i \gamma_i$, the grafting $\text{gr}_\lambda X$ defined by this extension procedure agrees with the obvious generalization of grafting along simple closed curves, wherein the geodesics $\gamma_1, \dots, \gamma_n$ are simultaneously replaced with cylinders.

For a general measured lamination $\lambda \in \mathcal{ML}(S)$, one can think of $\text{gr}_\lambda X$ as a Riemann surface obtained from X by thickening the leaves of the lamination λ in a manner dictated by the transverse measure. This intuition is made precise by the definition of a canonical stratification of $\text{gr}_\lambda X$ in the next section.

4.2 Thurston's Theorem

Projective grafting is a universal construction—every projective structure can be obtained from it, and in exactly one way:

Theorem 4.1 (Thurston [unpublished]). *The projective grafting map $\text{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ is a homeomorphism.*

The proof of Theorem 4.1 proceeds by explicitly constructing the inverse map Gr^{-1} using complex projective and hyperbolic geometry. We will now sketch this construction; details can be found in [64].

The embedded case. First suppose that $Z \in \mathcal{P}(S)$ is a projective surface whose developing map is an embedding (an *embedded projective structure*). The image of the developing map is a domain $\Omega \subset \mathbb{CP}^1$ invariant under the action of $\pi_1(S)$ by the holonomy representation ρ . In this case, we will describe the inverse of projective grafting in terms of convex hulls in hyperbolic space. See [32] for details on these hyperbolic constructions.

Considering \mathbb{CP}^1 as the ideal boundary of hyperbolic space \mathbb{H}^3 , let $\text{Pl}(Z)$ denote the boundary of the hyperbolic convex hull of $(\mathbb{CP}^1 - \Omega)$. Then $\text{Pl}(Z)$ is a *convex pleated plane* in \mathbb{H}^3 invariant under the action of $\pi_1(S)$ by isometries.

When equipped with the path metric, the pleated plane $\text{Pl}(Z)$ is isometric to \mathbb{H}^2 , and by this isometry, the action of $\pi_1(S)$ on \mathbb{H}^3 corresponds to a discontinuous cocompact action on \mathbb{H}^2 . Let $Y \in \mathcal{T}(S)$ denote the marked quotient surface.

The pleated plane $\text{Pl}(Z)$ consists of totally geodesic pieces (*plaques* or *facets*) meeting along geodesic *bending lines*. Applying the isometry $\text{Pl}(Z) \simeq \mathbb{H}^2$ to the union of the bending lines yields a geodesic lamination, which has a natural transverse measure recording the amount of bending of $\text{Pl}(Z)$. The lamination and measure are $\pi_1(S)$ -invariant, and therefore descend to the quotient, defining an element $\lambda \in \mathcal{ML}(S)$.

Thus, starting from an embedded projective structure Z , we obtain a hyperbolic structure Y and a measured lamination λ . To show that we have inverted the projective grafting map, we must check that $\text{Gr}_\lambda Y = Z$.

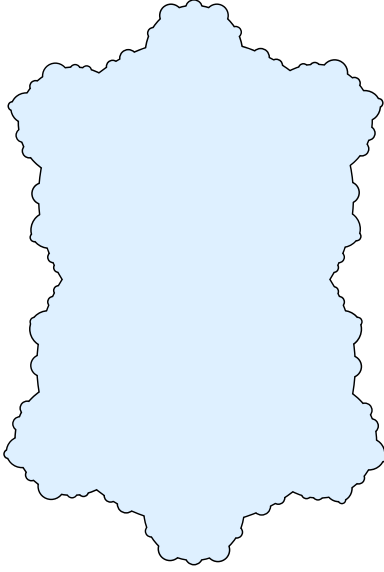
Nearest-point projection. There is a *nearest-point projection map* $\kappa : \Omega \rightarrow \text{Pl}(Z)$ that sends $z \in \Omega$ to the first point on $\text{Pl}(Z)$ that is touched by an expanding family of horoballs in \mathbb{H}^3 based at z . Convexity of $\text{Pl}(Z)$ ensures that this point is well-defined. In fact, from each $z \in \Omega$ we obtain not just a nearest point on $\text{Pl}(Z)$, but also a *support plane* H_z which contains $\kappa(z)$ and whose normal vector at that point defines a geodesic ray with ideal endpoint z . This gives a map $\hat{\kappa} : \Omega \rightarrow \mathcal{H}^{2,1}$, where $\mathcal{H}^{2,1}$ is the space of planes in \mathbb{H}^3 (the *de Sitter space*).

The *canonical stratification* of Ω is the decomposition into fibers of the map $\hat{\kappa}$. Strata are of two types:

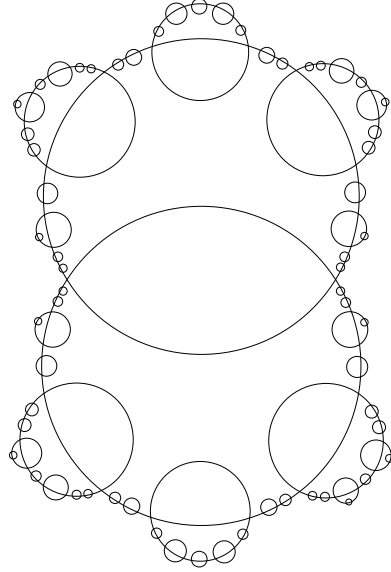
- 1-dimensional strata—circular arcs that map homeomorphically by κ onto bending lines of $\text{Pl}(Z)$, and
- 2-dimensional strata—regions with nonempty interior bounded by circular arcs which map homeomorphically by κ to the totally geodesic pieces of $\text{Pl}(Z)$.

If λ is supported on a single closed geodesic (or on a finite union of them), the 1-dimensional strata and the boundary geodesics of the 2-dimensional strata in $\Omega \simeq \tilde{Z}$ fill out a collection of lunes, and the interiors of the 2-dimensional strata correspond by κ to the complementary regions of the lift of λ , realized geodesically on Y , to $\tilde{Y} \simeq \mathbb{H}^2$. See Figure 3 for an example of this type. This is the arrangement of lunes and circular polygons giving the projective structure of $\text{Gr}_\lambda Y$, and so $Z = \text{Gr}_\lambda Y$. A limiting argument shows the same holds for general λ .

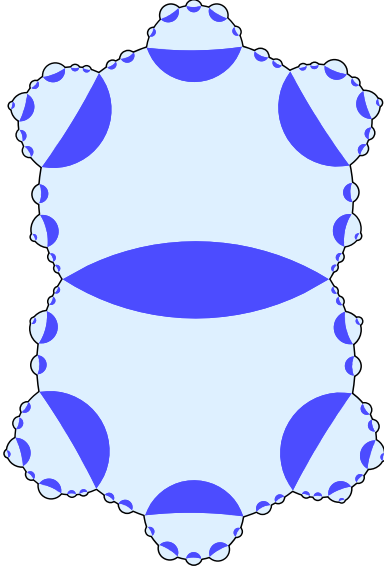
The general case. The key to inverting the projective grafting in the embedded case is the construction of the convex pleated plane $\text{Pl}(Z)$. For general



(a) Developed image



(b) Maximal disks



(c) Lunes and ideal polygons

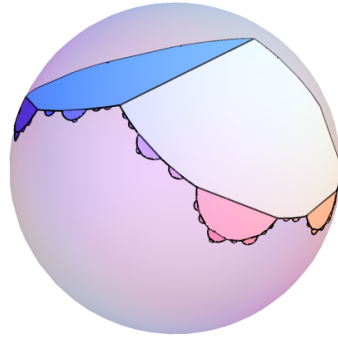
(d) Pleated plane in \mathbb{H}^3 (Klein model)

Figure 3. Four views of a projective structure lifted to the universal cover of a surface. The example shown here is an approximation of an embedded structure on a surface of genus 2, obtained by grafting along a separating simple closed curve. The approximation includes only a few of the maximal disks.

$Z \in \mathcal{P}(S)$, this is replaced by a *locally convex pleated plane* defined using the projective geometry of Z itself, rather than its developed image.

Let $f : \tilde{Z} \rightarrow \mathbb{CP}^1$ be the developing map of Z . A *round disk* in \tilde{Z} is an open subset U such that f is injective on U and $f(U)$ is an open disk in \mathbb{CP}^1 . The round disks in \tilde{Z} are partially ordered with respect to inclusion. A maximal element for this ordering is a *maximal round disk*.

Each maximal round disk U in \tilde{Z} corresponds to a disk in \mathbb{CP}^1 , and thus to an oriented plane H_U in \mathbb{H}^3 . Allowing U to vary over all maximal round disks in \tilde{Z} gives a family of oriented planes, and the envelope of this family is a locally convex pleated plane $\text{Pl}(Z)$.

The rest of the convex hull construction generalizes as follows: The intrinsic geometry of $\text{Pl}(Z)$ is hyperbolic, with quotient Y , and the bending of $\text{Pl}(Z)$ is recorded by a measured lamination λ . In place of the nearest-point projection and support planes, we have a *collapsing map* $\kappa : \tilde{Z} \rightarrow \text{Pl}(Z)$ and a *co-collapsing map* $\hat{\kappa} : \tilde{Z} \rightarrow \mathcal{H}^{2,1}$ (see also [25, §2, §7]). The fibers of $\hat{\kappa}$ induce a canonical stratification of \tilde{Z} , and separating the 1- and 2-dimensional strata describes Z as the projective grafting $\text{Gr}_\lambda Y$.

Note that the canonical stratification of \tilde{Z} is $\pi_1(S)$ -invariant, and therefore we have a corresponding decomposition of Z into 1- and 2-dimensional pieces. We will also refer to this as the canonical stratification. Similarly, the collapsing map descends to a map $\kappa : Z \rightarrow Y$ between quotient surfaces, which sends the union of 1-dimensional strata and boundary geodesics of 2-dimensional strata onto the bending lamination $\lambda \subset Y$.

The canonical stratification for complex projective structures is discussed further in [64, §1.2], where it is also generalized to n -manifolds equipped with *flat conformal structure* (see also [78] [102]).

Dual trees. When grafting along a simple closed curve γ with weight t , each bending line of the associated pleated plane in \mathbb{H}^3 has a one-parameter family of support planes (see Figure 4). These give an interval in the image of $\hat{\kappa}$, and the angle between support planes gives a metric on this interval, making it isometric to $[0, t] \subset \mathbb{R}$. Alternatively, this metric could be defined as the restriction of the Lorentzian metric of $\mathcal{H}^{2,1}$, where the restriction is positive definite because any pair of support planes of a given bending line intersect (see [102, §5] [78, §3, §6.5]).

The intervals corresponding to different bending lines meet at vertices corresponding to support planes of flat pieces. This gives $\hat{\kappa}$ the structure of a metric tree, the *dual tree* of the weighted curve $t\gamma$, denoted $T_{t\gamma}$. As this notation suggests, this tree depends only on $t\gamma$ (through the bending lines, their bending angles, and the adjacency relationship between bending lines and flat pieces) and not on the quotient hyperbolic structure of the pleated plane. The equivariance of the pleated plane with respect to $\pi_1(S)$ determines an isometric action of $\pi_1(S)$ on $T_{t\gamma}$.

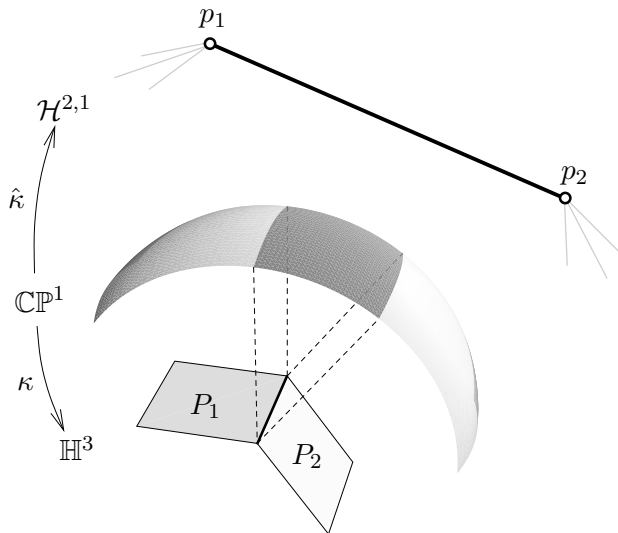


Figure 4. A lune between two maximal disks collapses to a bending line between two planes (P_1, P_2) , and co-collapses to an interval between two points (p_1, p_2) .

For a general grafting lamination $\lambda \in \mathcal{ML}(S)$, the image of $\hat{\kappa}$ has the structure of a \mathbb{R} -tree (see [101, Ch. 9] [78, §6, §11]), a geodesic metric space in which each pair of points is joined by a unique geodesic which is isometric to an interval in \mathbb{R} [90, Ch. 2]. This *dual \mathbb{R} -tree of λ* , denoted T_λ , is also equipped with an isometric action of $\pi_1(S)$.

4.3 The Thurston metric

We have seen that when grafting along a simple closed curve, the resulting projective surface $\text{Gr}_{t_\gamma} X$ has a natural conformal metric that combines the hyperbolic structure of X and the Euclidean structure of the cylinder. This is the *Thurston metric* (or *projective metric*) on the projective surface.

This definition can be extended to arbitrary projective surfaces by taking limits of the metrics obtained from an appropriate sequence of simple closed curves; however, we will prefer an intrinsic description of the metric based on complex projective geometry.

Kobayashi construction. The Kobayashi metric on a complex manifold is defined by a norm on each tangent space, where the length of a vector v is the infimum of lengths given to it by holomorphically immersed disks (each of which is equipped with its hyperbolic metric). For a surface Z with a projective

structure, there is a variant of the Kobayashi metric in which one minimizes length over the smaller class of *projectively immersed disks*, that is, immersions $\Delta \rightarrow \tilde{Z}$ that are locally Möbius with respect to the projective structure on Δ as a subset of \mathbb{CP}^1 . The resulting “projective Kobayashi metric” is the *Thurston metric* of Z [110, §2.1].

Relation to grafting. This intrinsic definition of the Thurston metric is related to grafting as follows: for each $z \in \tilde{Z}$, there is a unique maximal round disk $U \subset \tilde{Z}$ such that the (lifted) Thurston metric at z agrees with the hyperbolic metric on U . Furthermore, the set of points in \tilde{Z} that correspond to a given maximal disk U is a stratum in the canonical stratification of Z . Thus the Thurston metric is built from 2-dimensional hyperbolic regions with geodesic boundary and 1-dimensional geodesic strata. For $Z = \text{Gr}_\lambda X$, the union of the hyperbolic strata covers a subset of Z isometric to $(X - \lambda)$, where λ is realized geodesically on the hyperbolic surface X . When $\lambda = t\gamma$ is supported on a simple closed curve, the 1-dimensional strata sweep out Euclidean strips in $\widetilde{\text{Gr}_{t\gamma} X}$, which cover a Euclidean cylinder in $\text{Gr}_{t\gamma} X$, recovering the synthetic description of the Thurston metric in this case.

Conformal metrics and regularity. The Thurston metric on Z is a nondegenerate Riemannian metric compatible with the underlying complex structure $\pi(Z)$, i.e. it is a *conformal metric* on the Riemann surface. In local complex coordinates, the line element of such a metric has the form $\rho(z)|dz|$, where $\rho(z)$ is the real-valued *density function*. In the case of simple grafting, the density function of the Thurston metric is smooth on the hyperbolic and Euclidean pieces, but it is only C^1 on the interface between them. (The discontinuity in its second derivative is necessary since the curvature changes along the interface.) In general, the Thurston metric of a projective surface is $C^{1,1}$, meaning that its density function has Lipschitz derivatives, with Lipschitz constant locally bounded on $\mathcal{P}(S)$ [78].

Variation of metrics. The Thurston metric is a continuous function of the projective structure $Z \in \mathcal{P}(S)$ with respect to the topology of locally uniform convergence of density functions: For a sequence $Z_n \rightarrow Z \in \mathcal{P}(S)$, the Lipschitz bound on the derivatives of the Thurston metrics shows that uniform convergence follows from pointwise convergence, which in turn follows from the locally uniform convergence of the developing maps $f_n : \Delta \rightarrow \mathbb{CP}^1$ (or from the continuous variation of the associated locally convex pleated surfaces).

Area. A conformal metric on a Riemann surface with density function ρ induces an area measure by integration of $\rho^2 = \rho(z)^2 |dz|^2$.

The total area of $\text{Gr}_\lambda X$ with respect to the Thurston metric is $4\pi(g-1) + \ell(\lambda, X)$, where $\ell(\lambda, X)$ is the length of the measured lamination λ with respect to the hyperbolic metric of X . The two terms correspond to the two types of strata: The union of the 2-dimensional strata has area $4\pi(g-1)$, because

it is isometric to the complement of a geodesic lamination (a null set) in the hyperbolic surface X . The union of the 1-dimensional strata has area $\ell(\lambda, X)$, which is the continuous extension to $\mathcal{ML}(S)$ of the function $t\ell(\gamma, X)$ giving the area of the Euclidean cylinder $\gamma \times [0, t]$ in the case of simple grafting.

Curvature. The Gaussian curvature K and curvature 2-form Ω of a smooth conformal metric are related to its density function ρ by

$$\begin{aligned} K &= -\frac{1}{\rho^2} \Delta \log \rho \\ \Omega &= K \rho^2 = -\Delta \log \rho. \end{aligned} \tag{4.1}$$

In particular, such a metric has nonpositive Gaussian curvature if and only if $\log \rho$ is a subharmonic function.

The Thurston metric is not smooth everywhere, but it is nonpositively curved (NPC), meaning that its geodesic triangles are thinner than triangles in Euclidean space with the same edge lengths. As in the smooth case, this implies that $\log \rho$ is subharmonic, so we have a nonpositive measure $\Omega = -\Delta \log(\rho)$ that generalizes the curvature 2-form [99] (see also [98] [55] [54] [87]). For the Thurston metric, Ω is absolutely continuous, $\Omega = K \rho^2$ where K is the (a.e. defined) Gaussian curvature function. By a generalization of the Gauss-Bonnet theorem, the total mass of Ω (which is the integral of K) is $-4\pi(g-1)$ [54].

Hyperbolic and Euclidean. Since the Gaussian curvature of the Thurston metric is -1 in the interior of each 2-dimensional stratum, and these have total area $4\pi(g-1)$, the curvature of the Thurston metric is almost everywhere 0 in the union of the 1-dimensional strata. In this sense, grafting along a general lamination can be seen as the operation of inserting a Euclidean “surface” in place of a geodesic lamination, generalizing the case of closed leaves.

4.4 Conformal grafting maps

Having discussed the projective grafting construction and its inverse, we turn our attention to properties of the conformal grafting map $\text{gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{T}(S)$.

Using techniques from the theory of harmonic maps between surfaces (see §6.3), Tanigawa showed that this map is proper when either one of the coordinates is fixed:

Theorem 4.2 (Tanigawa [110]). *For each $\lambda \in \mathcal{ML}(S)$, the λ -grafting map $\text{gr}_\lambda : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ is a proper smooth map. For each $X \in \mathcal{T}(S)$, the X -grafting map $\text{gr}_X : \mathcal{ML}(S) \rightarrow \mathcal{T}(S)$ is a proper continuous map.*

Properness allows global properties of these maps to be derived from local considerations. For example, Scannell and Wolf showed that the λ -grafting map is an immersion, and therefore it is a local diffeomorphism. Since a proper local diffeomorphism is a covering map, this result and Theorem 4.2 give:

Theorem 4.3 (Scannell-Wolf [103]). *For each $\lambda \in \mathcal{ML}(S)$, the λ -grafting map $\text{gr}_\lambda : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ is a diffeomorphism.*

Earlier, Tanigawa had shown that gr_λ is a diffeomorphism when $\lambda \in \mathcal{ML}(S)$ is supported on a finite set of simple closed curves with weights that are integral multiples of 2π [110]. This follows from Theorem 4.2 because holonomy considerations (see §5) imply that gr_λ is a local diffeomorphism in this case.

In the general case, Scannell and Wolf analyze the Thurston metric and conformal grafting map through the interaction of two differential equations: The Liouville equation, which relates a Riemannian metric to its curvature, and the Jacobi equation, which determines the variation of a geodesic with respect to a family of Riemannian metrics. Analytic estimates for these equations are used to show that a 1-parameter family of graftings $t \mapsto \text{gr}_\lambda X_t$ cannot be conformally equivalent to first order unless $(d/dt)X_t|_{t=0} = 0$, which gives injectivity of the derivative of gr_λ .

As a consequence of Theorem 4.3, for any $\lambda \in \mathcal{ML}(S)$ the set of projective structures with grafting lamination λ projects homeomorphically to $\mathcal{T}(S)$ by the forgetful map. That is, the set of such projective structures forms a smooth section $\sigma_\lambda : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ of π , which is given by

$$\sigma_\lambda(X) = \text{Gr}_\lambda(\text{gr}_\lambda^{-1}(X)). \quad (4.2)$$

Note that this is compatible with our previous definition of the standard Fuchsian structure $\sigma_0(X)$, since this is the unique projective structure on X with zero grafting lamination. As with Theorem 4.3, in the special case of 2π -integral weighted multicurves, the existence of these smooth sections follows from the earlier work of Tanigawa.

Fixing X and varying $\lambda \in \mathcal{ML}(S)$, we can also use Theorem 4.3 to parameterize the fiber $P(X)$; that is,

$$\lambda \mapsto \sigma_\lambda(X)$$

gives a homeomorphism $\mathcal{ML}(S) \rightarrow P(X)$ (compare [26, §4]). It is the inverse of the map which sends $\text{Gr}_\lambda Y \in P(X)$ to λ .

Building on the Scannell-Wolf result, the author and Wolf showed that the X -grafting map is also a local homeomorphism, leading to:

Theorem 4.4 (Dumas and Wolf [27]). *For each $X \in \mathcal{T}(S)$, the X -grafting map $\text{gr}_\bullet X : \mathcal{ML}(S) \rightarrow \mathcal{T}(S)$ is a homeomorphism. Furthermore, this homeomorphism is bitangentiable.*

The last claim in this theorem involves the regularity of the grafting map as λ is varied. Let $f : U \rightarrow V$ be a continuous map, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets. The *tangent map* of f at x , denoted $T_x f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, is defined by

$$T_x f(v) = \lim_{\epsilon \rightarrow 0^+} \frac{f(x + \epsilon v) - f(x)}{\epsilon}.$$

The map f is *tangentiable* if this limit exists for all $(x, v) \in U \times \mathbb{R}^n$, and if the convergence is locally uniform in v when x is fixed. Intuitively, a tangentiable map is one which has one-sided derivatives everywhere. These notions generalize naturally to maps between smooth manifolds (e.g. $\mathcal{T}(S)$), piecewise linear manifolds (e.g. $\mathcal{ML}(S)$), or manifolds defined by an atlas of charts with tangentiable transition functions. Tangentiable maps and manifolds are discussed in [10].

A homeomorphism f is called *bitangentiable* if both f and f^{-1} are tangentiable, and if every tangent map of f or f^{-1} is a homeomorphism. Thus a bitangentiable homeomorphism is the analogue of a diffeomorphism in the tangentiable category.

The connection between grafting, projective structures, and tangentiability was studied by Bonahon, following work of Thurston on the infinitesimal structure of the space $\mathcal{ML}(S)$ [115]; the fundamental result, which strengthens Thurston's theorem, is

Theorem 4.5 (Bonahon [10]). *The projective grafting map $\text{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ is a bitangentiable homeomorphism.*

The proof of Theorem 4.4 uses Theorem 4.3, the above result of Bonahon, and a further complex linearity property of the tangent map of projective and conformal grafting (see [10] [9, §10], also [27, §3]). This complex linearity provides a “duality” between variation of $\text{gr}_\lambda X$ under changes in X and λ ; in a certain sense, grafting behaves like a holomorphic function, where X and λ are the real and imaginary parts of its parameter, respectively. This allows infinitesimal injectivity of $\text{gr}_\bullet X_0$ near λ_0 to be derived from the infinitesimal injectivity of gr_{λ_0} near X_0 .

After applying some additional tangentiable calculus, this infinitesimal injectivity is converted to local injectivity of $\text{gr}_\bullet X$, from which Theorem 4.4 follows by properness (Theorem 4.2).

5 Holonomy

We now turn our attention to the holonomy representations of projective structures in relation to the grafting and Schwarzian coordinate systems for $\mathcal{P}(S)$. General references for these matters include [47] [46] [28] [51] [36].

5.1 Representations and characters

Let $\mathcal{R}(S) = \text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{C}))$ denote the set of homomorphisms (representations) from $\pi_1(S)$ to $\text{PSL}_2(\mathbb{C})$, which is of an affine \mathbb{C} -algebraic variety (as a subset of $(\text{PSL}_2(\mathbb{C}))^N \simeq (\text{SO}_3(\mathbb{C}))^N$). The group $\text{PSL}_2(\mathbb{C})$ acts algebraically on $\mathcal{R}(S)$ by conjugation, and there is a quotient *character variety*

$$\mathcal{X}(S) = \mathcal{R}(S) // \text{PSL}_2(\mathbb{C})$$

in the sense of geometric invariant theory. Concretely, the points of $\mathcal{X}(S)$ are in one-to-one correspondence with the set of *characters*, i.e. \mathbb{C} -valued functions on $\pi_1(S)$ of the form

$$\gamma \mapsto \text{tr}^2(\rho(\gamma))$$

where $\rho \in \mathcal{R}(S)$. Mapping a character to its values on an appropriate finite subset of $\pi_1(S)$ gives an embedding of $\mathcal{X}(S)$ as an affine variety in \mathbb{C}^n . See [49] for a discussion of $\text{PSL}_2(\mathbb{C})$ character varieties, building on the work of Culler-Shalen in the $\text{SL}_2(\mathbb{C})$ case [20]. Algebraic and topological properties of character varieties are also studied in [44, §9] [41] [97].

Liftability. The variety $\mathcal{X}(S)$ splits into two irreducible components according to whether or not the associated representations lift from $\text{PSL}_2(\mathbb{C})$ to $\text{SL}_2(\mathbb{C})$ (see [41] [97]). Denote these by $\mathcal{X}_0(S)$ and $\mathcal{X}_1(S)$, where the former consists of liftable characters. Each of these components has complex dimension $6g - 6$, which agrees with the “expected dimension”, i.e. $6g - 6 = (\dim \text{PSL}_2(\mathbb{C}))(N_{\text{gens}} - N_{\text{relators}} - 1)$.

Elementary and non-elementary. When working with the character variety, complications may arise because the invariant-theoretic quotient $\mathcal{X}(S)$ is singular, or because it is not the same as the quotient set $\mathcal{R}(S)/\text{PSL}_2(\mathbb{C})$. However we can avoid most of these difficulties by restricting attention to a subset of characters (which contains those that arise from projective structures).

A representation $\rho \in \mathcal{R}(S)$ is *elementary* if its action on \mathbb{H}^3 by isometries fixes a point or an ideal point, or if it preserves an unoriented geodesic, otherwise it is *non-elementary*.

A non-elementary representation is determined up to conjugacy by its character, so there is a one-to-one correspondence between set set of conjugacy classes of non-elementary representations and the set $\mathcal{X}'(S) \subset \mathcal{X}(S)$ of characters of non-elementary representations.

The subset $\mathcal{X}'(S)$ is open and lies in the the smooth locus of the character variety [44] [46] [38]. Thus $\mathcal{X}'(S)$ is a complex manifold of dimension $6g - 6$, and is the union of the open and closed subsets $\mathcal{X}'_i(S) = \mathcal{X}'(S) \cap \mathcal{X}_i(S)$, $i = 1, 2$.

Fuchsian and quasi-Fuchsian spaces. The character variety $\mathcal{X}(S)$ contains the space $\mathcal{QF}(S)$ of conjugacy classes of quasi-Fuchsian representations of $\pi_1(S)$ as an open subset of $\mathcal{X}'_0(S)$. The parameterization of $\mathcal{QF}(S)$ by the pair of quotient conformal structures gives a holomorphic embedding

$$\mathcal{T}(S) \times \mathcal{T}(\bar{S}) \rightarrow \mathcal{X}(S),$$

where \bar{S} represents the surface S with the opposite orientation (see [85, §4.3]). In this embedding, the diagonal $\{(X, \bar{X}) \mid X \in \mathcal{T}(S)\}$ corresponds to the set $\mathcal{F}(S)$ of Fuchsian representations, giving an identification $\mathcal{F}(S) \simeq \mathcal{T}(S)$. Note that this is *not* a holomorphic embedding of Teichmüller space into the character variety; the image is a totally real submanifold.

5.2 The holonomy map

Since the holonomy representation $\rho \in \mathcal{R}(S)$ of a projective structure Z is determined up to conjugacy, the associated character $[\rho] \in \mathcal{X}(S)$ is uniquely determined. Considering $[\rho]$ as a function of Z gives the *holonomy map*

$$\text{hol} : \mathcal{P}(S) \rightarrow \mathcal{X}(S).$$

In fact, the image of hol lies in $\mathcal{X}'_0(S)$: A lift to $\text{SL}_2(\mathbb{C})$ is given by the linear monodromy of the Schwarzian ODE (3.1). The holonomy representation is non-elementary because S does not admit an affine or spherical structure; for details, see [5, pp. 297-304] [63, Thm. 3.6] [46, §2] [43, Thm. 19, Cor. 3].

Holonomy theorem. For hyperbolic structures on compact manifolds, the holonomy representation determines the geometric structure. For projective structures on surfaces, the same is true *locally*:

Theorem 5.1 (Hejhal [47], Earle [28], Hubbard [51]). *The holonomy map $\text{hol} : \mathcal{P}(S) \rightarrow \mathcal{X}(S)$ is a local biholomorphism.*

Originally, Hejhal showed that the holonomy map is a local homeomorphism using a cut-and-paste argument. Earle and Hubbard gave alternate proofs of this result, along with differential calculations showing that the map is locally biholomorphic. Recall that when considering $\mathcal{P}(S)$ as a complex manifold, we are using the complex structure induced by the quasi-Fuchsian sections.

A more general holonomy theorem for (G, X) structures is discussed in [40].

Negative results. Despite the simple local behavior described by Theorem 5.1, the global behavior of the holonomy map is quite complicated:

Theorem 5.2.

- (1) *The holonomy map is not injective. In fact, all of the fibers of the holonomy map are infinite.*
- (2) *The holonomy map is not a covering of its image.*

The non-injectivity in (1) follows from the discussion of 2π -grafting in §5.4 below. Hejhal established (2) by showing that the path lifting property of coverings fails for the holonomy map [47]. The infinite fibers of the holonomy map arise from the existence of *admissible curves* that can be used to alter a projective structure while preserving its holonomy [36] [66, Ch. 7]; this is similar to the “constructive approach” discussed in §5.5 below.

Further pathological behavior of the holonomy map is discussed in [59, §5].

Surjectivity. Of course one would like to know which representations arise from the holonomy of projective structures. We have seen that in order to arise from a projective structure, a character must be non-elementary and liftable (i.e. $\text{hol}(\mathcal{P}(S)) \subset \mathcal{X}'_0(S)$). These necessary conditions are also sufficient:

Theorem 5.3 (Gallo, Kapovich, and Marden [36]). *Every non-elementary liftable $\text{PSL}_2(\mathbb{C})$ -representation of $\pi_1(S)$ arises from the holonomy of a projective structure on S . Equivalently, we have $\text{hol}(\mathcal{P}(S)) = \mathcal{X}'_0(S)$.*

In the same paper it is also shown that the non-elementary non-liftable representations arise from *branched* projective structures. In both cases, the developing map of a projective structure with holonomy representation ρ is constructed by gluing together simpler projective surfaces that can be analyzed directly. A key technical result that enables this construction is:

Theorem 5.4 ([36]). *Let $\rho : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$ be a homomorphism with non-elementary image. Then there exists a pants decomposition of S such that the restriction of ρ to any component of the decomposition is a marked rank-2 classical Schottky group. In particular, the image of every curve in the decomposition is loxodromic.*

Projective structures on pairs of pants with loxodromic boundary holonomy are analyzed in [36, §§6-7].

Holonomy deformations. We have seen that projective structures on a Riemann surface X form an affine space modeled on $Q(X)$ (§3.2). Thus, given a non-elementary representation $\rho \in \mathcal{X}'_0(S)$, projective structures provide deformations of ρ as follows: Find $Z \in \mathcal{P}(S)$ with $\text{hol}(Z) = \rho$, which is possible by Theorem 5.3, and consider the family of holonomy representations $\{\text{hol}(Z + \phi) \mid \phi \in Q(X)\}$. This gives a holomorphic embedding of \mathbb{C}^{3g-3} into $\mathcal{X}(S)$, a family of *projective deformations* of ρ . (Compare [74] [75], where Kra

refers to a projective structure on X as a *deformation* of the Fuchsian group uniformizing X .)

These deformations could be compared with the classical quasi-conformal deformation theory of Kleinian groups. Projective deformations are especially interesting because they are insensitive to the discreteness of the image of a representation, and because they apply to quasiconformally rigid Kleinian groups. On the other hand, it is difficult to describe the global behavior of a projective deformation explicitly, and there is often no canonical choice for the preimage of ρ under the holonomy map.

5.3 Holonomy and bending

The holonomy map for projective structures is related to the grafting coordinate system through the notion of *bending deformations*. We now describe these deformations, mostly following Epstein and Marden [32]). In doing so, we are essentially re-creating the projective grafting construction of §4.1 while working entirely in hyperbolic 3-space, and starting with a Fuchsian representation rather than a hyperbolic surface.

Bending Fuchsian groups. We begin with an algebraic description of bending. A primitive element $\gamma \in \pi_1(S)$ representing a simple closed curve that separates the surface S determines a \mathbb{Z} -amalgamated free product decomposition

$$\pi_1(S) = \pi_1(S_1) *_{\langle \gamma \rangle} \pi_1(S_2)$$

where $(S - \gamma) = S_1 \sqcup S_2$. Note that the representative γ determines an orientation of the closed geodesic, and using this orientation, we make the convention that S_2 lies to the right of the curve. Given a homomorphism $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ and an element $A \in \mathrm{PSL}_2(\mathbb{C})$ centralizing $\rho(\gamma)$, there is a deformed homomorphism ρ' uniquely determined by

$$\rho'(x) = \begin{cases} \rho(x) & \text{if } x \in \pi_1(S_1) \\ A\rho(x)A^{-1} & \text{if } x \in \pi_1(S_2). \end{cases} \quad (5.1)$$

Similarly, a nonseparating curve γ corresponds to a presentation of $\pi_1(S)$ as an HNN extension, and again each centralizing element gives a deformation of ρ . See [42, §3] for further discussion of this deformation procedure.

When ρ is a Fuchsian representation and A is an elliptic element having the same axis as $\rho(\gamma)$, the homomorphism ρ' is a *bending deformation* of ρ . When A rotates by angle t about the axis of $\rho(\gamma)$, clockwise with respect to the orientation, we denote the deformed representation by $\beta_{t\gamma}(\rho) = \rho'$. Up to conjugacy, this deformation depends only on the angle t and the curve γ , not on the representative in $\pi_1(S)$ or the induced orientation.

The “bending” terminology refers to the geometry of the action of $\pi_1(S)$ on \mathbb{H}^3 by $\beta_{t\gamma}(\rho)$. The Fuchsian representation ρ preserves a plane $\mathbb{H}^2 \subset \mathbb{H}^3$, whereas we will see that the bending deformation $\beta_{t\gamma}(\rho)$ preserves a locally convex pleated (or *bent*) plane.

In terms of characters, the Fuchsian representation ρ_0 is a point in $\mathcal{F}(S) \simeq \mathcal{T}(S)$ and bending defines a map

$$\beta : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \rightarrow \mathcal{X}(S).$$

Like grafting, this map extends continuously to measured laminations [32, Thm. 3.11.5], giving

$$\beta : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{X}(S).$$

Note that while the bending path $t \mapsto \beta_{t\gamma}(X)$ is 2π -periodic, there is no apparent periodicity when bending along a general measured lamination.

Earthquakes and quakebends. The centralizer of a hyperbolic Möbius transformation $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ contains all of the elliptic and hyperbolic transformations with the same axis as γ , but in defining bending we have only considered the elliptic transformations. The deformation corresponding (by formula (5.1)) to a pure translation is known as an *earthquake*, and the common generalization of a bending or earthquake deformation (corresponding to the full centralizer) is a *quakebend* or *complex earthquake*. For further discussion of these deformations, see [32] [114] [86].

Bending cocycles. An alternate definition of the bending deformation makes the geometric content of the construction more apparent. Realize the simple closed curve γ as a hyperbolic geodesic on the surface $X \in \mathcal{T}(S)$, and consider the full preimage $\tilde{\gamma} \subset \mathbb{H}^2$ of γ in the universal cover; thus $\tilde{\gamma}$ consists of infinitely many complete geodesics, the *lifts* of γ . By analogy with the terminology for a pleated plane in \mathbb{H}^3 , the connected components of $\mathbb{H}^2 - \tilde{\gamma}$ will be called *plaques*. For the purposes of this discussion we regard \mathbb{H}^2 as a plane in \mathbb{H}^3 , stabilized by $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$.

Given $x, y \in (\mathbb{H}^2 - \tilde{\gamma})$, let (g_1, \dots, g_n) be the set of lifts of γ that separate x from y , ordered according to the way they intersect the oriented geodesic segment from x to y , with g_1 closest to x . Orient each geodesic g_i so that y lies to the right. For any $t \in \mathbb{R}$, define the *bending cocycle* $B(x, y) \in \mathrm{PSL}_2(\mathbb{C})$ by

$$B(x, y) = E(g_1, t)E(g_2, t) \cdots E(g_n, t),$$

where $E(g, t)$ is an elliptic Möbius transformation with fixed axis g and clockwise rotation angle t .

In case x and y lie in a facet, this empty product is understood to be the identity. This construction defines a map $B : (\mathbb{H}^2 - \tilde{\gamma}) \times (\mathbb{H}^2 - \tilde{\gamma}) \rightarrow \mathrm{PSL}_2(\mathbb{C})$. Clearly we have $B(x, x) = I$ and $B(x, y)$ only depends on the

plaques containing x and y . Furthermore, the map B satisfies the *cocycle relation*

$$B(x, y)B(y, z) = B(x, z) \quad \text{for all } x, y, z \in \mathbb{H}^2 - \tilde{\gamma}, \quad (5.2)$$

and the *equivariance relation*

$$B(\gamma x, \gamma y) = \rho_0(\gamma)B(x, y)\rho_0(\gamma)^{-1} \quad \text{for all } \gamma \in \pi_1(S) \quad (5.3)$$

where $\rho_0 \in \mathcal{F}(S)$ represents Y .

The connection between the bending cocycle and the bending deformation described above is as follows (compare [32, Lem. 3.7.1]).

Lemma 5.5. *Given $Y \in \mathcal{T}(S)$, a simple closed curve γ , and $t \in \mathbb{R}$, choose a basepoint $O \in (\mathbb{H}^2 - \tilde{\gamma})$ and define*

$$\rho(\gamma) = B(O, \gamma O)\rho_0(\gamma),$$

where $\rho_0 \in \mathcal{F}(S)$ represents Y and B is the bending cocycle associated to Y , γ , and t . Then ρ is a homomorphism, and it lies in the same conjugacy class as the bending deformation $\beta_{t\gamma}(Y)$.

In other words, the bending cocycle records the “difference” between a Fuchsian character ρ_0 and the deformed character $\beta_{t\gamma}(\rho_0)$. The bending cocycle and this lemma extend naturally to measured laminations [32, §3.5.3].

Bending and grafting. The key observation relating bending and grafting is that the bending deformation $\beta_\lambda(Y) : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$ preserves the locally convex pleated plane in \mathbb{H}^3 with intrinsic hyperbolic structure Y and bending lamination λ . In exploring this connection, let us suppose that $\lambda = t\gamma$ is supported on a simple closed curve. The pleating map $\text{Pl} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ can be defined in terms of the bending cocycle as

$$\text{Pl}(x) = B(O, x)x$$

where as before $O \in (\mathbb{H}^2 - \tilde{\gamma})$ is a base point. Equivariance of this map with respect to $\pi_1(S)$ then follows from Lemma 5.5 and the properties (5.2)-(5.3) of the bending cocycle. As written, this pleating map is only defined on $\mathbb{H}^2 - \tilde{\gamma}$, however it extends continuously to \mathbb{H}^2 because on the two sides of a lift $g \subset \tilde{\gamma}$, the values of $B(O, \cdot)$ differ by an elliptic Möbius transformation that fixes g pointwise.

The same reasoning shows that the image of Pl is a locally convex pleated plane: Since B is locally constant away from $\tilde{\gamma}$, the plaques map into planes in \mathbb{H}^3 , and when two such plaques share a boundary geodesic g , the images of the plaques in \mathbb{H}^3 meet along a geodesic $\text{Pl}(g)$ with bending angle t (which is to say, their enveloping planes are related by an elliptic Möbius transformation fixing their line of intersection, with rotation angle t).

We have seen that the holonomy of the projective structure $Z = \text{Gr}_\lambda Y$ also preserves the equivariant pleated plane in \mathbb{H}^3 constructed by bending $Y \simeq \mathbb{H}^2$ along λ . This leads to the fundamental relationship between grafting, bending and the holonomy map (see [86, §2]):

$$\text{hol}(\text{Gr}_\lambda Y) = \beta_\lambda(Y). \quad (5.4)$$

For laminations supported on simple closed curves, this is simply the observation that the processes of inserting lunes into $\mathbb{H} \subset \mathbb{CP}^1$ (which gives projective grafting) and bending $\mathbb{H}^2 \subset \mathbb{H}^3$ along geodesics (which gives the bending deformation) are related to one another by the convex hull construction of §4.2. The general equality follows from this case by continuity of hol , Gr , and β .

Using (5.4) we can think of projective grafting as a “lift” of the bending map $\beta : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{X}(S)$ through the locally diffeomorphic holonomy map $\text{hol} : \mathcal{P}(S) \rightarrow \mathcal{X}(S)$ (which is not a covering).

5.4 Fuchsian holonomy

Let $\mathcal{P}_\mathcal{F}(S) = \text{hol}^{-1}(\mathcal{F}(S))$ denote the set of all projective structures with Fuchsian holonomy.

We can construct examples of projective structures in $\mathcal{P}_\mathcal{F}(S)$ using grafting. Because of the 2π -periodicity of bending along a simple closed geodesic γ , the projective structures $\{\text{Gr}_{2\pi n\gamma} Y \mid n \in \mathbb{N}\}$ all have the same Fuchsian holonomy representation ρ_0 (up to conjugacy), which is the representation uniformizing Y . Of course $n = 0$ gives the standard Fuchsian structure on Y .

For $n > 0$ these projective structures have underlying Riemann surfaces of the form $\text{gr}_{2\pi n\gamma} Y$, and due to the 2π -lunes inserted in the projective grafting construction, their developing maps are surjective. This construction of “exotic” Fuchsian projective structures is due independently to Maskit [83], Hejhal [47, Thm. 4], and Sullivan-Thurston [109].

Goldman’s classification. Let $\mathcal{ML}_\mathbb{Z}(S)$ denote the countable subset of $\mathcal{ML}(S)$ consisting of disjoint collections of simple closed geodesics with positive integral weights. Generalizing the case of a single geodesic, every projective structure of the form $\text{Gr}_{2\pi\lambda} Y$ with $\lambda \in \mathcal{ML}_\mathbb{Z}(S)$ has Fuchsian holonomy. Goldman showed that all Fuchsian projective structures arise in this way:

Theorem 5.6 (Goldman [39]). *Let $Z \in \mathcal{P}_\mathcal{F}(S)$ and let $Y = \mathbb{H}^2/\text{hol}(Z)(\pi_1(S))$ be the hyperbolic surface associated to the holonomy representation. Then $Z = \text{Gr}_{2\pi\lambda} Y$ for some $\lambda \in \mathcal{ML}_\mathbb{Z}$.*

In terms of the holonomy map $\text{hol} : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$, this result shows that we can identify $\mathcal{P}_\mathcal{F}(S)$ with countably many copies of Teichmüller space,

$$\text{Gr}^{-1} : \mathcal{P}_\mathcal{F}(S) \xrightarrow{\simeq} (2\pi\mathcal{ML}_\mathbb{Z}(S)) \times \mathcal{T}(S),$$

and the restriction of the holonomy map to any one of these spaces $\{2\pi\lambda\} \times \mathcal{T}(S)$ gives the natural isomorphism $\mathcal{T}(S) \simeq \mathcal{F}(S)$.

Alternatively, using Theorem 4.3 in combination with Theorem 5.6, we can characterize $\mathcal{P}_{\mathcal{F}}(S)$ as the union of countably many sections of π ,

$$\mathcal{P}_{\mathcal{F}}(S) = \bigcup_{\lambda \in \mathcal{ML}_{\mathbb{Z}}} \sigma_{2\pi\lambda}(\mathcal{T}(S)).$$

Note the difference between these two descriptions of $\mathcal{P}_{\mathcal{F}}(S)$: In describing it as a union of sections, we see that the intersection of $\mathcal{P}_{\mathcal{F}}(S)$ with a fiber $P(X) = \pi^{-1}(X)$ consists of a countable discrete set naturally identified with $\mathcal{ML}_{\mathbb{Z}}(S)$, whereas in the holonomy picture we describe the intersection of $\mathcal{P}_{\mathcal{F}}(S)$ with $\text{hol}^{-1}(Y)$ in similar terms.

Describing $\mathcal{P}_{\mathcal{F}}(S)$ as a union of the smooth sections $\sigma_{2\pi\lambda}(\mathcal{T}(S))$ of π also allows us to conclude that each intersection between $\mathcal{P}_{\mathcal{F}}(S)$ and a fiber $P(X)$ is transverse. Previously, Faltings established this transversality result in the greater generality of *real holonomy*, that is, the projective structures in $\text{hol}^{-1}(\mathcal{X}_{\mathbb{R}}(S))$ where $\mathcal{X}_{\mathbb{R}}(S) \subset \mathcal{X}(S)$ consists of real-valued characters of homomorphisms of $\pi_1(S)$ into $\text{PSL}_2(\mathbb{C})$.

Theorem 5.7 (Faltings [33]). *Let $Z \in P(X)$ be a projective structure with real holonomy. Then $\text{hol}(P(X))$ is transverse to $\mathcal{X}_{\mathbb{R}}(S)$ at $\text{hol}(Z)$.*

The characters in $\mathcal{X}_{\mathbb{R}}(S)$ correspond to homomorphisms that are conjugate into $\text{SU}(2)$ or $\text{PSL}_2(\mathbb{R})$. Both cases include many non-Fuchsian characters, as a homomorphism $\rho : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{R})$ is Fuchsian if and only if its Euler class is maximal, $e(\rho) = 2g - 2$. Goldman describes the projective structures with real holonomy in terms of grafting in [39, §2.14] [40, pp. 14-15].

5.5 Quasi-Fuchsian holonomy

Let $\mathcal{P}_{\mathcal{QF}}(S) = \text{hol}^{-1}(\mathcal{F}(S))$ denote the set of all projective structures with quasi-Fuchsian holonomy, which is an open subset of $\mathcal{P}(S)$.

Goldman's proof of Theorem 5.6 involves a study of the topology and geometry of developing maps of Fuchsian projective structures. The topological arguments apply equally well to projective structures with quasi-Fuchsian holonomy, and the information they provide can be summarized as follows:

Theorem 5.8 (Goldman [39]). *Let $Z \in \mathcal{P}_{\mathcal{QF}}(S)$ have developing map $f : \tilde{Z} \rightarrow \mathbb{CP}^1$, and let $\Lambda \subset \mathbb{CP}^1$ be the limit set of the holonomy group, a Jordan curve with complementary regions Ω_{\pm} . Then:*

- (1) *The quotient of the developing preimage of the limit set, denoted $\Lambda(Z) = f^{-1}(\Lambda)/\pi_1(S)$, consists of a finite collection of disjoint simple closed curves.*

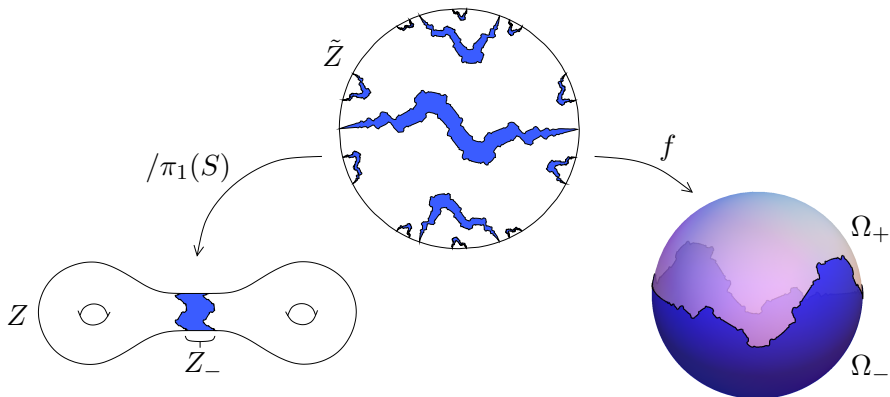


Figure 5. The relationship between the developing map f and the domains of discontinuity Ω_{\pm} for a projective structure with quasi-Fuchsian holonomy. In this example, the open set Z_- is an annulus, so the wrapping invariant is a simple closed curve with unit weight.

- (2) *The quotient of the developing preimage of Ω_- , denoted $Z_- = f^{-1}(\Omega_-)/\pi_1(S)$ consists of a finite collection of disjoint homotopically essential annuli bounded by the curves in $\Lambda(Z)$. In particular, the curves in $\Lambda(Z)$ are naturally grouped into isotopic pairs.*

Recall that among the two domains of discontinuity, Ω_+ is distinguished by the fact that the orientation of its quotient marked Riemann surface agrees with that of S , while that of the quotient of Ω_- is opposite.

The topology of a typical (surjective) quasi-Fuchsian developing map is represented schematically in Figure 5.

Wrapping invariant. Given this description of the preimage of the limit set, there is a natural \mathbb{Z} -weighted multicurve associated to a quasi-Fuchsian projective structure Z : Suppose the collection of annuli Z_- represents homotopy classes $\gamma_1, \dots, \gamma_n$, and that there are n_i parallel annuli homotopic to γ_i . Define the *wrapping invariant*

$$\text{wr}(Z) = \sum_i n_i \gamma_i \in \mathcal{ML}_{\mathbb{Z}}(S).$$

Note that we could have also defined this using the family of curves $\Lambda(Z)$, since $2n_i$ is the number of parallel curves homotopic to γ_i .

Theorem 5.6 is derived from Theorem 5.8 by showing that for a Fuchsian projective structure Z , we have

$$Z = \mathrm{Gr}_{2\pi \mathrm{wr}(Z)} Y,$$

where Y is the quotient of \mathbb{H}^2 by the holonomy group, as above. In other words, for Fuchsian projective structures, the wrapping invariant is the grafting lamination (up to a multiple of 2π).

Quasi-Fuchsian components. Because limit sets vary continuously in $\mathcal{QF}(S)$, the wrapping invariant is a locally constant function on $\mathcal{P}_{\mathcal{QF}}(S)$. Thus $\mathcal{P}_{\mathcal{QF}}(S)$ breaks into countably many subsets

$$\mathcal{P}_{\mathcal{QF}}(S) = \bigcup_{\lambda \in \mathcal{ML}_{\mathbb{Z}}(S)} \mathcal{P}_{\lambda}(S), \quad \text{where } \mathcal{P}_{\lambda}(S) = \mathrm{wr}^{-1}(\lambda).$$

We will refer to these as *components* of $\mathcal{P}_{\mathcal{QF}}(S)$.

The quasi-Fuchsian component with zero wrapping invariant, $\mathcal{P}_0(S)$, consists of *standard* quasi-Fuchsian structures. The holonomy map gives a diffeomorphism

$$\mathrm{hol} : \mathcal{P}_0(S) \rightarrow \mathcal{QF}(S),$$

where the inverse map associates to $\rho \in \mathcal{QF}(S)$ the induced projective structure on the quotient $\Omega_+/\rho(\pi_1(S))$ of one domain of discontinuity. The developing map of a standard quasi-Fuchsian projective structure is a Riemann map $f : \mathbb{H} \xrightarrow{\cong} \Omega_+$.

The other components $\mathcal{P}_{\lambda}(S)$, with $\lambda \neq 0$, consist of *exotic* quasi-Fuchsian projective structures; as in the Fuchsian case, these have surjective developing maps. Unlike the Fuchsian case, however, the components $\mathcal{P}_{\lambda}(S)$ do not have a simple description in terms of the grafting coordinates on $\mathcal{P}(S)$. Nevertheless, when restricted to one of these components, the holonomy map

$$\mathrm{hol} : \mathcal{P}_{\lambda}(S) \rightarrow \mathcal{QF}(S),$$

is again a diffeomorphism. The inverse $\mathcal{QF}(S) \rightarrow \mathcal{P}_{\lambda}(S)$ can be constructed by either of two methods:

- (1) **Constructive approach.** In a generalization of 2π -integral projective grafting, one starts with a standard quasi-Fuchsian projective structure Z and glues annuli into the surface to produce a new projective structure which has the same holonomy but which has wrapping invariant λ . Allowing the starting structure to vary gives a map $\mathcal{QF}(S) \simeq \mathcal{P}_0(S) \rightarrow \mathcal{P}_{\lambda}(S)$ that is inverse to hol . See [39, §1.2] [60, §2.4] [66, Ch. 7] for details.
- (2) **Deformation approach.** Starting with a fixed Fuchsian representation $\rho_0 \in \mathcal{F}(S)$, any quasi-Fuchsian representation ρ can be obtained by a ρ_0 -equivariant quasiconformal deformation. By pulling back the

quasiconformal deformation through a developing map, one can simultaneously deform a Fuchsian projective structure Z_0 with holonomy ρ_0 to obtain a quasi-Fuchsian structure Z with holonomy ρ . This deformation does not change the wrapping invariant, so starting with $Z_0 = \text{Gr}_{2\pi\lambda}X$ and considering all quasiconformal deformations gives the desired map $\mathcal{QF}(S) \rightarrow \mathcal{P}_\lambda(S)$. See [107, §3][57, §2.5].

Thus the structure of $\mathcal{P}_{\mathcal{QF}}(S)$ is similar to that of $\mathcal{P}_{\mathcal{F}}(S)$ described above: It consists of countably many connected components $\mathcal{P}_\lambda(S)$, each of which is diffeomorphic to $\mathcal{QF}(S)$ by the holonomy map (compare [66, §7.2] [57, §§2.5-2.6]).

Bumping of quasi-Fuchsian components. We say that two components $\mathcal{P}_\lambda(S)$ and $\mathcal{P}_\mu(S)$ *bump* if their closures intersect, i.e. if $\overline{\mathcal{P}_\lambda(S)} \cap \overline{\mathcal{P}_\mu(S)} \neq \emptyset$; an element of the intersection is called a *bumping point*. A component $\mathcal{P}_\lambda(S)$ *self-bumps* at $Z \in \mathcal{P}(S)$ if $U \cap \mathcal{P}_\lambda(S)$ is disconnected for all sufficiently small neighborhoods U of Z . These terms are adapted from similar phenomena in the theory of deformation spaces of Kleinian groups (surveyed in [16], see also [3] [4] [15] [50]).

The bumping of quasi-Fuchsian components has been studied by McMullen [86], Bromberg-Holt [14], and Ito [57] [60]. The basic problem of determining which component pairs bump is resolved by:

Theorem 5.9 (Ito [60]).

- (1) For any $\lambda, \mu \in \mathcal{ML}_{\mathbb{Z}}(S)$, the components $\mathcal{P}_\lambda(S)$ and $\mathcal{P}_\mu(S)$ bump.
- (2) For any $\lambda \in \mathcal{ML}_{\mathbb{Z}}(S)$, the component $\mathcal{P}_\lambda(S)$ self-bumps at a point in $\overline{\mathcal{P}_0(S)}$.

The bumping points constructed in the proof of this theorem are all derived from a construction of Anderson-Canary that illustrates the difference between algebraic and geometric convergence for Kleinian groups [3]. This construction was first applied to projective structures by McMullen to give an example of bumping between $\mathcal{P}_\lambda(S)$ and $\mathcal{P}_0(S)$ [86]. The holonomy representations for these bumping examples have accidental parabolics but are not quasiconformally rigid; recently, Brock, Bromberg, Canary, and Minsky have shown that these conditions are necessary for bumping [12] (compare [92]).

5.6 Discrete holonomy

Let $\mathcal{D}(S) \subset \mathcal{X}(S)$ denote the set of characters of discrete representations, and let $\mathcal{P}_{\mathcal{D}}(S)$ denote the set of projective structures with discrete holonomy. Since $\mathcal{F}(S) \subset \mathcal{QF}(S) \subset \mathcal{D}(S)$, we have corresponding inclusions

$$\mathcal{P}_{\mathcal{F}}(S) \subset \mathcal{P}_{\mathcal{QF}}(S) \subset \mathcal{P}_{\mathcal{D}}(S).$$

Because hol is a local diffeomorphism, topological properties of $\mathcal{D}(S)$ correspond to those of $\mathcal{P}_{\mathcal{D}}(S)$. For example, $\mathcal{D}(S)$ is closed (see [61] [19]), and its interior is the set $\mathcal{QF}(S)$ of quasi-Fuchsian representations [108] [7]. Thus $\mathcal{P}_{\mathcal{D}}(S)$ is a closed subset of $\mathcal{P}(S)$ with interior $\mathcal{P}_{\mathcal{QF}}(S)$.

If $Z \in \mathcal{P}_{\mathcal{D}}(S)$ has holonomy ρ , then the associated pleated plane $\text{Pl}(Z) : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ is invariant under the holonomy group $\Gamma = \rho(\pi_1(S))$ and descends to a *locally convex pleated surface* in the quotient hyperbolic manifold $M = \mathbb{H}^3/\Gamma$:

$$\begin{array}{ccc} \mathbb{H}^2 & \xrightarrow{\text{Pl}(Z)} & \mathbb{H}^3 \\ \downarrow / \pi_1(S) & & \downarrow / \Gamma \\ Y & \longrightarrow & M \end{array}$$

Here $Y \in \mathcal{T}(S)$ is the hyperbolic surface such that $Z = \text{Gr}_{\lambda} Y$ for some $\lambda \in \mathcal{ML}(S)$.

The pleated surface arising from a projective structure Z with discrete holonomy may be one of the connected components of the boundary of the *convex core* of the associated hyperbolic manifold M . If so, the projective surface Z is the component of the ideal boundary of M on the “exterior” side of the pleated surface. Conversely, the ideal boundary and convex core boundary surfaces in a complete hyperbolic manifold are related by grafting (see [103, §5.1] [86, §2.8]).

For more general projective structures with discrete holonomy, the pleated surface need not be embedded in the quotient manifold, however it must lie within the convex core (see [17, §5.3.11]).

In addition to the Fuchsian and quasi-Fuchsian cases described above, projective structures with other classes of discrete holonomy representations have found application in Kleinian groups and hyperbolic geometry. For example, projective structures with degenerate holonomy are used in Bromberg’s approach to the Bers density conjecture [13], and those with Schottky holonomy are used in Ito’s study of sequences of Schottky groups accumulating on Bers’ boundary of Teichmüller space [58].

5.7 Holonomy in fibers

In contrast to the complicated global properties of the holonomy map, its restriction to a fiber is very well-behaved:

Theorem 5.10. *For each $X \in \mathcal{T}(S)$, the restriction $\text{hol}|_{P(X)}$ is a proper holomorphic embedding, whose image $\text{hol}(P(X))$ is a complex-analytic subvariety of $\mathcal{X}(S)$.*

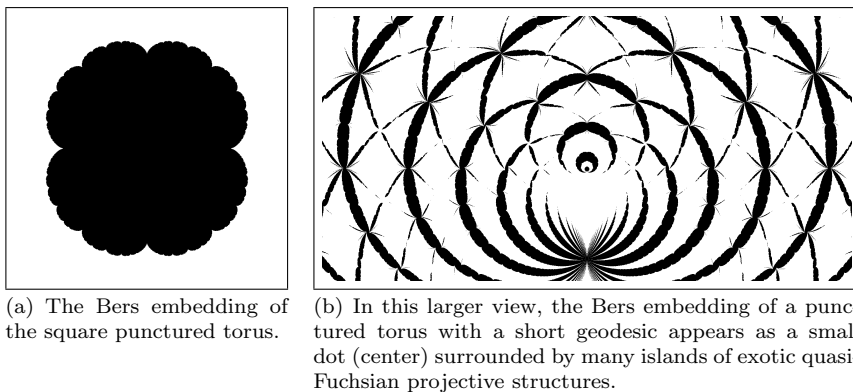


Figure 6. Islands of quasi-Fuchsian holonomy in $P(X) \simeq \mathbb{C}$ (where X is a punctured torus) exhibit complicated structure at small and large scales. These images were created using the software package *Bear* [24].

As stated, this theorem incorporates several related but separate results: Working in the context of systems of linear ODE on a fixed Riemann surface, Poincaré showed that the holonomy map is injective [5, p. 310] (see also [76] [47, Thm. 15]). Gallo, Kapovich, and Marden showed that the image is a complex-analytic subvariety [36], following an outline given by Kapovich [65]; when combined with injectivity, this implies properness. Tanigawa gave a more geometric argument establishing properness of $\text{hol}|_{P(X)}$ when considered as a map into the space $\mathcal{X}'(S)$ of non-elementary characters [111]. Tanigawa's argument relies on the existence of loxodromic pants decompositions (Theorem 5.4), which was announced in [65] and proved in [36].

Fuchsian and quasi-Fuchsian holonomy in fibers. For any $X \in \mathcal{T}(S)$, let $P_{\mathcal{D}}(X) = P(X) \cap \mathcal{P}_{\mathcal{D}}(S)$ denote the set of projective structures with discrete holonomy and with underlying complex structure X . Similarly, we define $P_{\mathcal{QF}}(X)$ and $P_{\mathcal{F}}(X)$ as the subsets of $P(X)$ having quasi-Fuchsian and Fuchsian holonomy, respectively.

We have already seen (in §5.4) that the $P_{\mathcal{F}}(X)$ consists of the countable discrete set of projective structures $\{\sigma_{2\pi\lambda}(X) \mid \lambda \in \mathcal{ML}_{2\pi\mathbb{Z}}(S)\}$. Since the holonomy map is continuous, and $\mathcal{QF}(S)$ is an open neighborhood of $\mathcal{F}(S)$ in $\mathcal{X}(S)$, each of these Fuchsian points has a neighborhood in $P(X)$ consisting of quasi-Fuchsian projective structures with the same wrapping invariant. Elements of $P_{\mathcal{F}}(X)$ are sometimes called *Fuchsian centers* (or *centers of grafting* [2]), because they provide distinguished center points within these “islands” of quasi-Fuchsian holonomy (see [26, §13] [82, Thm. 6.6.10]).

Using the Schwarzian parameterization, the intersection $\mathcal{P}_0(S) \cap P(X)$, consisting of the standard quasi-Fuchsian projective structures on X , can be

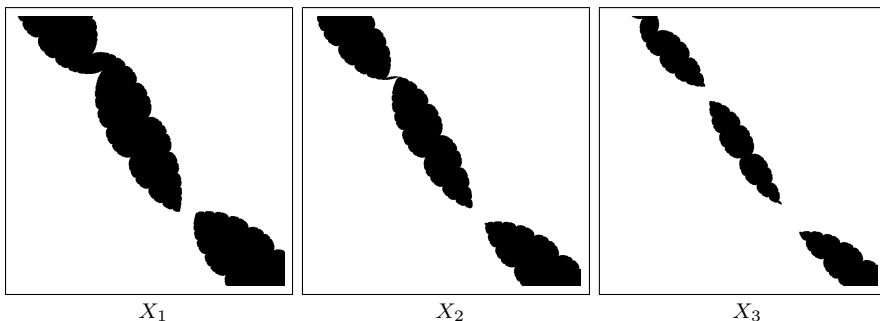


Figure 7. Islands of quasi-Fuchsian holonomy in $P(X)$ appear to break apart as the complex structure X is changed, suggesting that some islands do not contain Fuchsian centers. Each image shows a small square in $P(X_i) \simeq \mathbb{C}$, where $\{X_1, X_2, X_3\}$ are closely-spaced points in the Teichmüller space of the punctured torus.

considered as an open set $B_X \subset Q(X) \simeq \mathbb{C}^{3g-3}$. This set is the image of the holomorphic *Bers embedding* of Teichmüller space [106], and in particular it is connected and contractible. We also have $B(1/2) \subset B_X \subset B(3/2)$, where $B(r) = \{\phi \in Q(X) \mid \|\phi\|_\infty < r\}$, as a consequence of Nehari's theorem [91]. See Figure 6 for examples of Bers embeddings of the Teichmüller space of punctured tori.

For $\lambda \neq 0$, it is not known whether the set $\mathcal{P}_\lambda(S) \cap P(X)$ is connected (or bounded), though experimental evidence in the punctured case suggests that it often has many connected components, and that the structure of the connected components changes with X (see Figure 7). Of course, only one component contains the Fuchsian structure $\sigma_{2\pi\lambda}(X)$.

Quasi-Fuchsian versus discrete in a fiber. In the space of all projective structures, the quasi-Fuchsian structures form the interior of the set with discrete holonomy. The same relationship holds for $P_{\mathcal{QF}}(X)$ and $P_{\mathcal{D}}(X)$.

Theorem 5.11 (Shiga and Tanigawa [107], Matsuzaki [84]). *For any $X \in \mathcal{T}(S)$, we have $P_{\mathcal{QF}}(X) = \text{int}(P_{\mathcal{D}}(X))$.*

In comparing these sets, one inclusion is immediate: Since $\text{int}(\mathcal{P}_{\mathcal{D}}(S)) = \mathcal{P}_{\mathcal{QF}}(S)$, we have $\text{int}(P_{\mathcal{D}}(X)) \supset P_{\mathcal{QF}}(X)$. The opposite inclusion is more subtle. Each component of the interior of $P_{\mathcal{D}}(X)$ necessarily consists of quasiconformally conjugate, discrete, faithful representations without accidental parabolics. However there exist $(3g - 3)$ -dimensional holomorphic families of *singly degenerate* surface groups in $\mathcal{X}(S)$ which satisfy these conditions, but which are not quasi-Fuchsian. Such a family could account for an open subset

of $P_{\mathcal{D}}(X)$ (in either of two topologically distinct ways [84]), and a key step in the proof of the theorem is to exclude this possibility.

6 Comparison of parameterizations

6.1 Compactifications

Compactification of $\mathcal{ML}(S)$. The space of measured laminations has the structure of a cone: The group \mathbb{R}^+ acts by scaling the transverse measure ($\lambda \mapsto t\lambda$, $t \in \mathbb{R}^+$) and the empty lamination $0 \in \mathcal{ML}(S)$ is the unique fixed point of this action. The orbit of a nonzero lamination is a *ray* in $\mathcal{ML}(S)$. The space of rays,

$$\mathbb{P}\mathcal{ML}(S) = (\mathcal{ML}(S) - \{0\})/\mathbb{R}^+,$$

or *projective measured laminations* forms a natural boundary for $\mathcal{ML}(S)$. We say that a sequence $\lambda_i \in \mathcal{ML}(S)$ converges to $[\lambda] = \mathbb{R}^+ \cdot \lambda \in \mathbb{P}\mathcal{ML}(S)$ if there exists a sequence of positive real numbers c_i such that $c_i \rightarrow 0$ and $c_i \lambda_i \rightarrow \lambda$ in $\mathcal{ML}(S)$. The induced compactification

$$\overline{\mathcal{ML}(S)} = \mathcal{ML}(S) \cup \mathbb{P}\mathcal{ML}(S)$$

is homeomorphic to a closed ball, with interior $\mathcal{ML}(S) \simeq \mathbb{R}^{6g-6}$ and boundary $\mathbb{P}\mathcal{ML}(S) \simeq S^{6g-7}$. See [94, Ch. 3] for further discussion of the spaces $\mathcal{ML}(S)$ and $\mathbb{P}\mathcal{ML}(S)$, and [34] for related discussion of the space of measured foliations, which is naturally identified with $\mathcal{ML}(S)$ (as described in [80] [66, §11.8-11.9]).

Compactification of $\mathcal{T}(S)$. Recall that \mathcal{S} denotes the set of isotopy classes of simple closed curves on S , or equivalently, the simple closed geodesics of any hyperbolic structure on S . Thurston defined a compactification of $\mathcal{T}(S)$ using the hyperbolic length map

$$\begin{aligned} L : \mathcal{T}(S) &\rightarrow \mathbb{R}^{\mathcal{S}} \\ X &\mapsto (\ell(\gamma, X))_{\gamma \in \mathcal{S}}. \end{aligned}$$

This map is an embedding, as is its projectivization

$$\mathbb{P}L : \mathcal{T}(S) \rightarrow \mathbb{P}^+ \mathbb{R}^{\mathcal{S}} = (\mathbb{R}^{\mathcal{S}} - \{0\})/\mathbb{R}^+,$$

and in each case, a suitable finite subset of \mathcal{S} suffices to determine the image of a point. The boundary $\partial \mathbb{P}L(\mathcal{T}(S))$ coincides with the image of $\mathbb{P}\mathcal{ML}(S)$ under the projectivization of the embedding

$$\begin{aligned} \mathcal{ML}(S) &\rightarrow \mathbb{R}^{\mathcal{S}} \\ \lambda &\mapsto (i(\gamma, \lambda))_{\gamma \in \mathcal{S}} \end{aligned}$$

where $i(\lambda, \gamma)$ denotes the total mass of γ with respect to the transverse measure of λ . This gives the *Thurston compactification*

$$\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$$

which has the topology of a closed $(6g-6)$ -ball. Concretely, a sequence $X_n \rightarrow \infty$ in Teichmüller space converges to $[\lambda] \in \mathbb{P}\mathcal{ML}(S)$ if for every pair of simple closed curves $\alpha, \beta \in \mathcal{S}$ we have

$$\frac{\ell(\alpha, X_i)}{\ell(\beta, X_i)} \rightarrow \frac{i(\alpha, \lambda)}{i(\beta, \lambda)}$$

whenever the right hand side is well-defined (i.e. $i(\beta, \lambda) \neq 0$). A detailed discussion of the Thurston compactification can be found in [34, Exp. 7-8] (see also [112] [117] [8] [66, Ch. 11] [82, §5.9]).

Compactification of $Q(X)$. Since the vector space $Q(X)$ has an action of \mathbb{R}^+ by scalar multiplication, it supports a natural compactification analogous to that of $\mathcal{ML}(S)$; in this case, the boundary is the space of rays

$$\mathbb{P}^+Q(X) = (Q(X) - \{0\})/\mathbb{R}^+$$

and we obtain $\overline{Q(X)} = Q(X) \cup \mathbb{P}^+Q(X)$ which is homeomorphic to a closed ball.

6.2 Quadratic differentials and measured laminations

The Hubbard-Masur theorem. For any $X \in \mathcal{T}(S)$, there is a natural map

$$\Lambda : Q(X) \rightarrow \mathcal{ML}(S)$$

which is defined by a two-step procedure: First, a quadratic differential ϕ has an associated *horizontal foliation* $\mathcal{F}(\phi)$, a singular foliation on X which integrates the distribution of vectors $v \in TX$ such that $\phi(v) \geq 0$. This foliation is equipped with a transverse measure, induced by integration of $|\operatorname{Im} \sqrt{\phi}|$. In a local coordinate where $\phi = dz^2$, the foliation is induced by the horizontal lines in \mathbb{C} , with transverse measure $|dy|$. Zeros of ϕ correspond to singularities of the foliation, where three or more half-leaves emanate from a point. See e.g. [66, §5.3, §11.3] [37, §2.2, Ch. 11] for a discussion of quadratic differentials and their measured foliations.

Now lift the horizontal foliation of ϕ to the universal cover $\tilde{X} \simeq \mathbb{H}^2$. Each non-singular leaf of the lifted foliation is a uniform quasi-geodesic, so it is a bounded distance from unique hyperbolic geodesic. The hyperbolic geodesics obtained in this way—the *straightening* of \mathcal{F} —form the lift of a geodesic lamination on X , and the transverse measure of the foliation induces a transverse measure on this lamination in a natural way [80]. The result is a measured lamination $\Lambda(\phi) \in \mathcal{ML}(S)$, which we call the *horizontal lamination* of ϕ .

The same constructions can be applied to the distribution of vectors satisfying $\phi(v) \leq 0$, which gives the *vertical foliation* and *vertical lamination* of ϕ . The former is induced by the foliation of \mathbb{C} by vertical lines in local coordinates such that $\phi = dz^2$. Note that multiplication by -1 in $Q(X)$ exchanges vertical and horizontal: for example, the horizontal lamination of $-\phi$ is the vertical lamination of ϕ .

The strong connection between quadratic differentials and measured laminations is apparent in:

Theorem 6.1 (Hubbard and Masur [53]). *For each $X \in \mathcal{T}(S)$, the map $\Lambda : Q(X) \rightarrow \mathcal{ML}(S)$ is a homeomorphism. In particular, every measured lamination is realized by a unique quadratic differential on X .*

Note that Hubbard and Masur work with measured foliations rather than measured laminations; the statement above incorporates the aforementioned straightening procedure to identify the two notions.

We call the inverse of Λ the *foliation map*, denoted $\phi_F : \mathcal{ML}(S) \rightarrow Q(X)$. Note that the definition of both Λ and ϕ_F depend on the choice of a fixed conformal structure X , but we suppress this dependence in the notation.

Since the transverse measure of $\Lambda(\phi)$ is obtained by integrating $|\operatorname{Im} \sqrt{\phi}|$, these maps have the following homogeneity properties:

$$\begin{aligned}\Lambda(c\phi) &= c^{\frac{1}{2}} \Lambda(\phi) \\ \phi_F(c\lambda) &= c^2 \phi_F(\lambda)\end{aligned}$$

for all $c \in \mathbb{R}^+$. Therefore Λ and ϕ_F descend to mutually inverse homeomorphisms between the spaces of rays $\mathbb{P}\mathcal{ML}(S)$ and $\mathbb{P}^+Q(X)$, and we also use Λ and ϕ_F to denote these induced maps.

Orthogonality and the antipodal map. Given $X \in \mathcal{T}(S)$, a pair of measured laminations $\lambda, \mu \in \mathcal{ML}(S)$ is *orthogonal with respect to X* if there exists $\phi \in Q(X)$ such that

$$\begin{aligned}\Lambda(\phi) &= \lambda \\ \Lambda(-\phi) &= \mu\end{aligned}$$

That is, λ and μ appear as the horizontal and vertical laminations of a single holomorphic quadratic differential on X . (Compare the torus case shown in Figure 8.)

By Theorem 6.1, two laminations λ and μ are orthogonal with respect to X if and only if

$$\phi_F(\lambda) = -\phi_F(\mu) \in Q(X).$$

Thus the homeomorphism $\phi_F : \mathcal{ML}(S) \rightarrow Q(X)$ turns orthogonal pairs into opposite quadratic differentials, and the set of X -orthogonal pairs is the graph

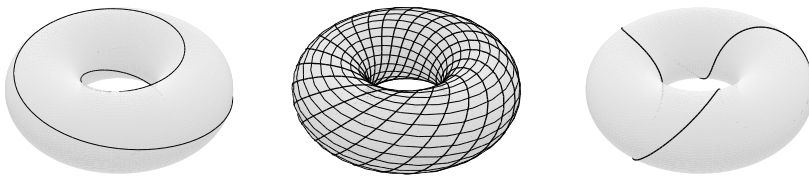


Figure 8. A pair of closed curves on a compact Riemann surface of genus 1 (as seen here on the far left and right) are “orthogonal” if they are isotopic to leaves of a pair of orthogonal geodesic foliations of the Euclidean metric (center). This situation is non-generic; typically, at least one of the two foliations will have dense leaves. For surfaces of higher genus, orthogonality of measured laminations is defined similarly, however there are many distinct singular Euclidean metrics.

of the *antipodal involution* $i_X : \mathcal{ML}(S) \rightarrow \mathcal{ML}(S)$ defined by

$$i_X(\lambda) = \Lambda(-\phi_F(\lambda)).$$

By homogeneity of Λ and ϕ_F , the antipodal map descends to $i_X : \mathbb{PM}\mathcal{L}(S) \rightarrow \mathbb{PM}\mathcal{L}(S)$. We say $[\lambda], [\mu] \in \mathbb{PM}\mathcal{L}(S)$ are orthogonal with respect to X if $i_X([\lambda]) = [\mu]$. See [25] for further discussion of the antipodal map and orthogonality.

6.3 Limits of fibers

Using the projective grafting homeomorphism $\text{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{P}(S)$, we can regard $\overline{\mathcal{ML}(S)} \times \overline{\mathcal{T}(S)}$ as a compactification of $\mathcal{P}(S)$. This is the *grafting compactification*.

Given $X \in \mathcal{T}(S)$, the fiber $P(X) \subset \mathcal{P}(S)$ corresponds to a set of pairs $\text{Gr}^{-1}(X) = \{(\lambda, Y) \mid \text{gr}_\lambda Y = X\}$ in the grafting coordinates. Since $P(X)$ is a distinguished subset of the Schwarzian parameterization of $\mathcal{P}(S)$, studying its behavior in the grafting parameterization is one way to study the relationship between these two coordinate systems. The asymptotic behavior of $P(X)$ can be described in terms of orthogonality:

Theorem 6.2 (Dumas [25]). *Let $(\lambda_n, Y_n) \in \mathcal{ML}(S) \times \mathcal{T}(S)$ be a divergent sequence such that $\text{Gr}_{\lambda_n} Y_n \in P(X)$ for all n . Then*

$$\lim_{n \rightarrow \infty} \lambda_n = [\lambda] \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} Y_n = i_X([\lambda]),$$

where these limits are taken in $\overline{\mathcal{ML}(S)}$ and $\overline{\mathcal{T}(S)}$, respectively.

In particular, the boundary of $P(X)$ in the grafting compactification of $\mathcal{P}(S)$ is the graph of the antipodal involution $i_X : \mathbb{PM}\mathcal{L}(S) \rightarrow \mathbb{PM}\mathcal{L}(S)$.

This theorem can be considered as evidence of compatibility between the grafting coordinates for $\mathcal{P}(S)$ and the foliation of $\mathcal{P}(S)$ by fibers of π . For example, we have:

Corollary 6.3. *The closure of $P(X)$ in $\overline{\mathcal{ML}(S)} \times \overline{\mathcal{T}(S)}$ is homeomorphic to a closed ball of dimension $6g - 6$.*

The proof of Theorem 6.2 in [25] is essentially a study of the collapsing and co-collapsing maps of a complex projective structure, and their relation to the harmonic maps variational problem. We now describe this variational technique, and then outline the main steps in the proof.

Harmonic maps. Let (M, g) and (N, h) be complete Riemannian manifolds, and assume that M is compact. If $f : M \rightarrow N$ is a smooth map, the *energy* of f is defined by

$$\mathcal{E}(f) = \frac{1}{2} \int_M \|df(x)\|_2^2 dg(x).$$

The map f is *harmonic* if it is a critical point of the energy functional. If N is also compact and h has negative sectional curvature, then any nontrivial homotopy class of maps $M \rightarrow N$ contains a harmonic map, and this map is an absolute minimum of the energy functional in the homotopy class [31]. Furthermore, the harmonic map is unique in its homotopy class, unless the image of M is a closed geodesic in N , in which case there is a 1-parameter family of harmonic maps obtained by rotation. General references for the theory of harmonic maps include [29] [30] [104], with particular applications to Teichmüller theory surveyed in [22].

Equivariant harmonic maps. If $\pi_1(M)$ acts by isometries on a Riemannian manifold \hat{N} , then we can define the energy of an equivariant map $\tilde{M} \rightarrow \hat{N}$ by integration of $\|df\|_2^2$ over a fundamental domain for the action of $\pi_1 M$ by deck transformations. This generalizes the energy of smooth maps $M \rightarrow N$, because the action of $\pi_1 M$ on \hat{N} need not have a Hausdorff quotient. Existence of harmonic maps is more delicate in this case, but can sometimes be recovered under additional restrictions on the group action. For example if M is a surface and $N = \mathbb{H}^3$ is equipped with the isometric action coming from a non-elementary representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$, then there is a unique equivariant harmonic map $h : \tilde{S} \rightarrow \mathbb{H}^3$ [23].

Singular targets. Korevaar and Schoen developed a deep generalization of the theory of harmonic maps in which the Riemannian manifold N is replaced by a nonpositively curved (NPC, also known as locally CAT(0)) metric space [72] [73] [71]. Here the energy functional is approximated by the average squared distance between the image of a point $x \in M$ and the image of a small sphere centered at x . Inequalities comparing distances in NPC spaces

to those in Euclidean space have an essential role in the development of this theory.

Generalizing the Riemannian case, we have the following equivariant existence and uniqueness results: If \hat{N} is a locally compact NPC space on which $\pi_1(M)$ acts by isometries without fixing any equivalence class of rays, then there is an equivariant harmonic map $h : \tilde{M} \rightarrow \hat{N}$, which is Lipschitz and energy-minimizing [73]. If furthermore \hat{N} is negatively curved (locally $\text{CAT}(\kappa)$, for some $\kappa < 0$), then the harmonic map is unique unless its image is a geodesic [89].

Harmonic maps from surfaces. When M is 2-dimensional, the energy functional depends only on the conformal class of the metric g , so it makes sense to consider harmonic maps from Riemann surfaces to Riemannian manifolds and nonpositively curved metric spaces. An important invariant of a harmonic map $f : X \rightarrow (N, h)$ from a Riemann surface is its *Hopf differential*

$$\Phi(f) = [f^*(h)]^{2,0} \quad (6.1)$$

which is a holomorphic quadratic differential. In the Riemannian case, the holomorphicity of $\Phi(f)$ is a consequence of the Euler-Lagrange equation of the energy functional [29, §10]. With a suitable generalization of the pullback metric (see [72, §2.3]), a holomorphic Hopf differential is also obtained from a harmonic map to a NPC metric space (compare [88, §5]).

We can use the same formula (6.1) to define a Hopf differential for any smooth map $X \rightarrow (N, h)$, which can be further generalized to maps with L^2 distributional derivatives, and to finite-energy maps to NPC metric spaces [72, Thm. 2.3.1]. The result is a L^1 measurable quadratic differential that is not necessarily holomorphic.

Harmonic maps and dual trees. Recall from §4.2 that for each $\lambda \in \mathcal{ML}(S)$ we have a dual \mathbb{R} -tree T_λ . This tree is a NPC metric space (even $\text{CAT}(\kappa)$ for all $\kappa < 0$) equipped with an isometric action of $\pi_1(S)$. The Hubbard-Masur construction of a quadratic differential on $X \in \mathcal{T}(S)$ with lamination λ can be described in terms of an equivariant harmonic map $X \rightarrow T_\lambda$.

Theorem 6.4 (Wolf [119], Daskalopoulos-Dostoglou-Wentworth [21]). *Let $h : \tilde{X} \rightarrow T_\lambda$ be an equivariant harmonic map to the dual \mathbb{R} -tree of $\lambda \in \mathcal{ML}(S)$. Then $\phi_F(\lambda) = -4\Phi(h)$.*

Harmonic maps and the Thurston compactification. The Thurston compactification of Teichmüller space can also be characterized in terms of Hopf differentials of harmonic maps from a fixed Riemann surface as follows:

Theorem 6.5 (Wolf [118]). *Fix $X \in \mathcal{T}(S)$ and let $Y_n \rightarrow \infty$ be a divergent sequence in $\mathcal{T}(S)$. Let $\Phi_n = \Phi(h_n)$ be the Hopf differential of the harmonic*

map $h_n : X \rightarrow Y_n$ compatible with the markings. Then

$$\Lambda(-\Phi_n) \rightarrow [\lambda] \in \mathbb{P}\mathcal{ML}(S) \text{ if and only if } Y_n \rightarrow [\lambda] \in \mathbb{P}\mathcal{ML}(S).$$

Collapsing, co-collapsing, and harmonic maps. Using the harmonic maps results presented above, we now describe the main steps of the proof of Theorem 6.2 in [25]. For simplicity, we will suppose that $\text{Gr}_{\lambda_n} Y_n \in P(X)$ and that both grafting coordinates have limits in $\mathbb{P}\mathcal{ML}(S)$, i.e.

$$\lim_{n \rightarrow \infty} \lambda_n = [\lambda] \quad \lim_{n \rightarrow \infty} Y_n = [\mu],$$

and we outline a proof that $i_X([\lambda]) = [\mu]$. The stronger statement of the theorem is derived from the same set of ideas.

Outline of proof of Theorem 6.2.

- (1) Both the collapsing maps $\kappa_n : X \rightarrow Y_n$ and the co-collapsing maps $\hat{\kappa}_n : \tilde{X} \rightarrow T_{\lambda_n}$ are *C-almost harmonic*, meaning that their energies exceed the minimum energies in their homotopy classes by at most C . Here C is a constant that depends only on the topology of S . (Compare [110].)
- (2) The maps κ_n and $\hat{\kappa}_n$ have an orthogonality relationship: their derivatives have rank 1 in the same subset of X (the Euclidean part of the Thurston metric), and in this set, the collapsed directions of κ_n and $\hat{\kappa}_n$ are orthogonal. This orthogonality relationship is expressed in terms of their Hopf differentials as

$$\Phi(\kappa_n) + \Phi(\hat{\kappa}_n) = 0. \tag{6.2}$$

- (3) Let $h_n : X \rightarrow Y_n$ and $\hat{h}_n : \tilde{X} \rightarrow T_{\lambda_n}$ denote the harmonic maps homotopic to κ_n and $\hat{\kappa}_n$, respectively. Then the projective limit of Hopf differentials $[\Phi] = \lim_{n \rightarrow \infty} \Phi(h_n)$ satisfies $[\Lambda(-\Phi)] = [\mu]$ by Theorem 6.5. Similarly, by Theorem 6.4, the projective limit $[\hat{\Phi}] = \lim_{n \rightarrow \infty} \Phi(\hat{h}_n) = \lim_{n \rightarrow \infty} (-\phi_F(\lambda_n)/4)$ satisfies $[\Lambda(-\hat{\Phi})] = [\lambda]$.
- (4) Since the pair of almost harmonic maps κ_n and $\hat{\kappa}_n$ have opposite Hopf differentials, one might expect that the associated harmonic maps h_n and \hat{h}_n have “almost opposite” Hopf differentials. Suppose that this is true in the sense of projective limits, i.e. that

$$[\Phi] = [-\hat{\Phi}] \in \mathbb{P}^+ Q(X). \tag{6.3}$$

Then we would have $[\Lambda(\Phi)] = [\lambda]$ and $[\Lambda(-\Phi)] = [\mu]$, or equivalently, that $i_X([\lambda]) = [\mu]$, completing the proof. Thus we need only derive (6.3).

- (5) The norm of the difference between the pullback metric of a C -almost harmonic map f to an NPC space and that of its homotopic harmonic map h is $O(C^{1/2}\mathcal{E}(h)^{1/2})$ as $\mathcal{E}(h) \rightarrow \infty$ (by an estimate of Korevaar and Schoen, see [72, §2.6]). Phrasing this in terms of Hopf differentials, which are the $(2, 0)$ parts of the pullback metrics, and using that $|\mathcal{E}(h) - 2\|\Phi(h)\|| = O(1)$, we have

$$\|\Phi(f) - \Phi(h)\|_1 \leq C'(1 + \|\Phi(h)\|_1^{\frac{1}{2}}).$$

In particular the norm of the difference is much smaller than either term as $\|\Phi(h)\| \rightarrow \infty$, and so the Hopf differentials of any sequence of C -almost harmonic maps with energy tending to infinity has the same projective limit as the Hopf differentials of the harmonic maps. Applying this to the collapsing and co-collapsing maps, and using (6.2), we have

$$[\Phi] = \lim_{n \rightarrow \infty} \Phi(\kappa_n) = \lim_{n \rightarrow \infty} (-\Phi(\hat{\kappa}_n)) = [-\hat{\Phi}],$$

and (6.3) follows. □

6.4 Limits of the Schwarzian

We now connect the previous discussion of asymptotics of grafting coordinates for $P(X)$ with the complex-analytic parameterization of $\mathcal{P}(S)$. Let $\overline{P(X)}$ denote the *Schwarzian compactification* of $P(X)$ obtained by attaching $\mathbb{P}^+Q(X)$ using the limiting behavior of the Schwarzian derivative, i.e. a sequence $Z_n \in P(X)$ converges to $[\phi]$ if $(Z_n - Z_0) \rightarrow [\phi]$ in the topology of $\overline{Q(X)}$. Here Z_0 denotes an arbitrary basepoint, which is used to identify $P(X)$ with $Q(X)$; the limit of a sequence in $\mathbb{P}^+Q(X)$ does not depend on this choice. Note that this construction only compactifies the individual fibers of $\mathcal{P}(S)$, but does not compactify $\mathcal{P}(S)$ itself.

There is a natural guess for the relationship between the Schwarzian compactification and the closure of $P(X)$ in the grafting compactification: The boundary of the latter is the set of X -antipodal pairs in $\mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(S)$, and each X -antipodal pair arises from a ray in the space of quadratic differentials, so one might expect a boundary point $[\phi] \in \mathbb{P}^+Q(X)$ to correspond to the pair consisting of its vertical and horizontal laminations. The following makes this intuition precise:

Theorem 6.6 (Dumas [26]). *The grafting and Schwarzian compactifications of $P(X)$ are naturally homeomorphic, and the boundary map $\mathbb{P}^+Q(X) \rightarrow \mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(S)$ is given by*

$$[\phi] \mapsto ([\Lambda(-\phi)], [\Lambda(\phi)]).$$

That is, for a divergent sequence in $P(X)$, the limit of the vertical (resp. horizontal) laminations of Schwarzian differentials is equal to the limit of the measured laminations (resp. hyperbolic structures) in the grafting coordinates.

This result about compactifications involves a comparison between two homeomorphisms $\mathcal{ML}(S) \rightarrow Q(X)$. One of these we have already seen—the foliation map ϕ_F which sends $\lambda \in \mathcal{ML}(S)$ to a quadratic differential whose horizontal foliation has straightening λ (§6.2). The other homeomorphism is derived from the Schwarzian parameterization of projective structures as follows. Recall (from §4.4) that there is a homeomorphism $\sigma_\bullet(X) : \mathcal{ML}(S) \rightarrow P(X)$ with the property that $\sigma_\lambda(X) \in P(X)$ is a projective structure with grafting lamination λ . Using $\sigma_0(X)$ as a basepoint, we compose with the Schwarzian parameterization $P(X) \simeq Q(X)$ to obtain the *Thurston map*:

$$\begin{aligned} \phi_T : \mathcal{ML}(S) &\rightarrow Q(X) \\ \lambda &\mapsto (\sigma_\lambda(X) - \sigma_0(X)) \end{aligned}$$

The Thurston map is a homeomorphism, and it satisfies $\phi_T(0) = 0$, but unlike the foliation map there is no *a priori* reason for ϕ_T to map rays in $\mathcal{ML}(S)$ to rays in $Q(X)$. However, the Thurston map does preserve rays in an asymptotic sense:

Theorem 6.7 ([26]). *For any $X \in \mathcal{T}(S)$, the foliation and Thurston maps are asymptotically proportional. Specifically, there exists a constant $C(X)$ such that*

$$\|\phi_F(\lambda) + 2\phi_T(\lambda)\|_1 \leq C(X) \left(1 + \|\phi_F(\lambda)\|_1^{\frac{1}{2}}\right)$$

for all $\lambda \in \mathcal{ML}(S)$.

Before discussing the proof of Theorem 6.7, we explain the connection with compactifications. In terms of the Thurston map, Theorem 6.6 asserts that if $\phi_T(\lambda_n) = \text{Gr}_{\lambda_n} Y_n$ is a divergent sequence in $P(X)$, then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_n &= \lim_{n \rightarrow \infty} \Lambda(-\phi_T(\lambda_n)) \in \overline{\mathcal{ML}(S)} \quad \text{and} \\ \lim_{n \rightarrow \infty} Y_n &= \lim_{n \rightarrow \infty} \Lambda(\phi_T(\lambda_n)) \in \overline{\mathcal{T}(S)}. \end{aligned} \tag{6.4}$$

Theorem 6.2 has already given a similar characterization in terms of the map ϕ_F ; we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_n &= \lim_{n \rightarrow \infty} \Lambda(\phi_F(\lambda_n)) \in \overline{\mathcal{ML}(S)} \quad \text{and} \\ \lim_{n \rightarrow \infty} Y_n &= \lim_{n \rightarrow \infty} \Lambda(-\phi_F(\lambda_n)) = \lim_{n \rightarrow \infty} i_X(\lambda_n) \in \overline{\mathcal{T}(S)}, \end{aligned} \tag{6.5}$$

where the first line is trivial since $\Lambda \circ \phi_F = \text{Id}$, and the second line follows from the definition of the antipodal map (§6.2). However, since ϕ_F and ϕ_T are

asymptotically proportional by a negative constant (Theorem 6.7), the limit characterizations (6.4) and (6.5) are equivalent, and Theorem 6.6 follows. See [26, §14] for details.

Thurston metrics and the Schwarzian. We now sketch the main ideas involved in the proof of Theorem 6.7. The proof is essentially a study of the Thurston metric on a complex projective surface (see §4.3). Recall that the goal is to show that $\|\phi_F(\lambda) + 2\phi_T(\lambda)\|_1 \leq C(X)\varepsilon(\lambda)$ where $\varepsilon(\lambda)$ is defined by

$$\varepsilon(\lambda) = 1 + \|\phi_F(\lambda)\|_1^{\frac{1}{2}}.$$

Outline of proof of Theorem 6.7.

- (1) The functions $\varepsilon(\lambda)$ and $\lambda \mapsto \|\phi_F(\lambda) + 2\phi_T(\lambda)\|_1$ are continuous on $\mathcal{ML}(S)$. Since weighted simple closed geodesics are dense in $\mathcal{ML}(S)$, it suffices establish an inequality relating these functions for such weighted geodesics, and the general case follows by continuity. Thus we will assume λ is a weighted simple closed geodesic for the rest of the proof.
- (2) Associated to such λ we have the following objects:
 - The Thurston metric ρ_λ of the projective structure $\sigma_\lambda(X) \in P(X)$
 - The decomposition $X = X_0 \sqcup X_{-1}$ of X into Euclidean and hyperbolic parts of ρ_λ . Here X_0 is an open cylinder, the union of the 1-dimensional strata in the canonical stratification.
 - The collapsing map $\kappa : X \rightarrow Y_\lambda = \text{gr}_\lambda^{-1}(X)$ and its Hopf differential $\Phi(\kappa)$, which is a measurable (non-holomorphic) quadratic differential supported on X_0 .
 - The ratio of conformally equivalent metrics ρ_λ/ρ_0 , a well-defined positive function on X . Here ρ_0 is the hyperbolic metric.
- (3) The Schwarzian derivative $\phi_T(\lambda)$ of the projective structure $\sigma_\lambda(X)$ decomposes as a sum of two terms,

$$\phi_T(\lambda) = -2\Phi(\kappa) + 2\mathcal{B}(\log(\rho_\lambda/\rho_0)), \quad (6.6)$$

where the second-order differential operator \mathcal{B} is defined by

$$\mathcal{B}(\eta) = [\text{Hess}(\eta) - d\eta \otimes d\eta]^{2,0}.$$

In this expression, the Hessian is computed using the hyperbolic metric ρ_0 . This decomposition follows from the cocycle property for a generalization of the Schwarzian derivative introduced by Osgood and Stowe [93].

- (4) The harmonic map estimate from the proof of Theorem 6.2 shows that the first term of the decomposition (6.6) is approximately proportional to $\phi_F(\lambda)$. Specifically, we have

$$\|\phi_F(\lambda) - 4\Phi(\kappa)\|_1 \leq C\varepsilon(\lambda). \quad (6.7)$$

Therefore it suffices to show that the L^1 norm of $\beta = \mathcal{B}(\log(\rho_\lambda/\rho_0))$ is also bounded by a multiple of $\varepsilon(\lambda)$.

- (5) By the definition of \mathcal{B} and the Cauchy-Schwartz inequality, the L^1 norm of β is bounded by the L^2 norms of the Hessian and gradient of $\log(\rho_\lambda/\rho_0)$ with respect to the hyperbolic metric. By standard elliptic theory, these are in turn bounded by the L^2 norms of $\log(\rho_\lambda/\rho_0)$ and its Laplacian.
- (6) The Laplacian of $\log(\rho_\lambda/\rho_0)$ is essentially the difference of the curvature 2-forms of ρ_λ and ρ_0 (compare (4.1) above, also [55]). For large grafting, the surface X is dominated by its Euclidean part, forcing most of the curvature of ρ_λ to concentrate near a finite set of points.
- (7) This curvature concentration phenomenon provides a bound for the norm $\|\Delta \log(\rho_\lambda/\rho_0)\|_{L^2(D)}$ on a hyperbolic disk $D \subset X$ of definite size. A bound on $\|\log(\rho_\lambda/\rho_0)\|_{L^2(D)}$ follows using a weak Harnack inequality, completing the local estimate $\|\beta\|_{L^1(D)} < C(X)$.
- (8) Finally, we make the local estimate global: If β were holomorphic, then we would have $\|\beta\|_{L^1(X)} \leq C'(X)\|\beta\|_{L^1(D)}$ by compactness of the unit sphere in $Q(X)$. While β is not holomorphic, the decomposition (6.6) and the estimate (6.7) show that β is close to a holomorphic quadratic differential, with difference of order $\varepsilon(\lambda)$. Combining this with the holomorphic case, we obtain $\|\beta\|_{L^1(X)} \leq C(X)\varepsilon(\lambda)$, completing the proof. \square

6.5 Infinitesimal compatibility

In this final section we discuss infinitesimal aspects of the map between the grafting and analytic coordinate systems for $\mathcal{P}(S)$.

The forgetful projection $\pi : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$ can be thought of as a coordinate function in the Schwarzian parameterization of $\mathcal{P}(S)$. The other “coordinate” in this parameterization is an element of the fiber $Q(X)$ of the bundle of quadratic differentials, but lacking a canonical trivialization for this bundle, there is no associated global coordinate map.

On the other hand, in the grafting coordinate system, we have a pair of well-defined coordinate maps $p_{\mathcal{ML}} : \mathcal{P}(S) \rightarrow \mathcal{ML}(S)$ and $p_{\mathcal{T}} : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$, which are defined by the property that the inverse of projective grafting is $\text{Gr}^{-1}(Z) = (p_{\mathcal{ML}}(Z), p_{\mathcal{T}}(Z)) \in \mathcal{ML}(S) \times \mathcal{T}(S)$.

The fiber of $p_{\mathcal{ML}}$ over λ consists of the projective structures $\{\text{Gr}_\lambda Y \mid Y \in \mathcal{T}(S)\}$. Since $\text{Gr}_\lambda : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$ is a smooth map, these fibers are smooth submanifolds of $\mathcal{P}(S)$.

The fiber of $p_{\mathcal{T}}$ over Y consists of the projective structures $\{\text{Gr}_\lambda Y \mid \lambda \in \mathcal{ML}(S)\}$. Bonahon showed that $\lambda \mapsto \text{Gr}_\lambda Y$ includes $\mathcal{ML}(S)$ into $\mathcal{P}(S)$ tangentially (see Theorem 4.5). However, the fibers of $p_{\mathcal{T}}$ have even more regularity than one might expect from this tangential parameterization:

Theorem 6.8 (Bonahon [10, Thm. 3, Lem. 13]). *For each $Y \in \mathcal{T}(S)$, the set $p_{\mathcal{T}}^{-1}(Y)$ is a C^1 submanifold of $\mathcal{P}(S)$.*

Compare [27, §4].

Note that each of the three coordinate maps $\pi, p_{\mathcal{ML}}, p_{\mathcal{T}}$ projects $\mathcal{P}(S)$ onto a space of half its real dimension, i.e. each has both range and fibers of real dimension $6g - 6$. Thus one might expect that for any two of these maps, the pair of fibers intersecting at a generic point $Z \in \mathcal{P}(S)$ would have transverse tangent spaces that span $T_Z \mathcal{P}(S)$. In fact, this is true at every point, and furthermore we have:

Theorem 6.9 (Dumas and Wolf [27]).

- (1) *The maps $\pi, p_{\mathcal{ML}}, p_{\mathcal{T}}$ have pairwise transverse fibers.*
- (2) *The fiber of any one of them projects homeomorphically by each of the others. Moreover, such a projection is a C^1 diffeomorphism whenever its range is $\mathcal{T}(S)$, and is a bitangentiable homeomorphism when the range is $\mathcal{ML}(S)$.*
- (3) *The product of any two of these maps gives a homeomorphism from $\mathcal{P}(S)$ to a product of two spaces of real dimension $6g - 6$.*

As before, we refer to Bonahon (see [10, §2]) for details about tangentiability, while limiting our focus to its geometric consequences. Also note that statement (1) of the theorem does *not* involve tangentiability, and only makes sense for fibers of $p_{\mathcal{T}}$ due to Theorem 6.8.

We sketch the proof of this theorem; the details we omit can be found in [27, Thms. 1.2, 4.1, 4.2, Cor. 4.3].

Sketch of proof of Theorem 6.9. Statement (3) follows because the inverse map for each pair of coordinates can be written explicitly in terms of Gr , gr_{λ} , and $\text{gr}_{\cdot}X$ and their inverses (which exist by Theorems 4.1, 4.3, and 4.4, respectively). For example, $p_{\mathcal{T}} \times \pi : \mathcal{P}(S) \rightarrow \mathcal{T}(S) \times \mathcal{T}(S)$ is a homeomorphism with inverse

$$(X, Y) \mapsto \text{Gr}_{(\text{gr}_{\cdot}X)^{-1}(Y)}X.$$

Similarly, the map $(\lambda, X) \mapsto \sigma_{\lambda}(X)$ is inverse to $p_{\mathcal{ML}} \times \pi$.

Statement (3) also shows that the restrictions of maps considered in statement (2) are homeomorphisms. To show that each case with target $\mathcal{T}(S)$ is actually a diffeomorphism, it is enough to show that the derivative of the restriction has no kernel (by the inverse function theorem). This kernel is the intersection of tangent spaces to fibers of two coordinate maps, thus this case will follow from statement (1). Similar reasoning applies in cases with target $\mathcal{ML}(S)$, where one deduces bitangentiability from transversality using a criterion of Bonahon [10, Lem. 4].

Thus the proof is reduced to the transversality statement (1), which has one case for each pair of coordinate maps. The pair $(p_{\mathcal{ML}}, p_{\mathcal{T}})$ follows easily from Thurston's theorem and the tangentiability of grafting (Theorems 4.1 and 4.5). For $(\pi, p_{\mathcal{ML}})$ or $(\pi, p_{\mathcal{T}})$, a vector in the intersection of tangent spaces lies in the kernel of a tangent map of either gr_{λ} or $\text{gr}_{\lambda} X$, which must therefore be zero, by Theorems 4.3 and 4.4. \square

References

- [1] L. Ahlfors and L. Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math. (2)*, 72:385–404, 1960.
- [2] C. Anderson. *Projective Structures on Riemann Surfaces and Developing Maps to \mathbb{H}^3 and \mathbb{CP}^n* . PhD thesis, University of California at Berkeley, 1998.
- [3] J. Anderson and R. Canary. Algebraic limits of Kleinian groups which rearrange the pages of a book. *Invent. Math.*, 126(2):205–214, 1996.
- [4] J. Anderson, R. Canary, and D. McCullough. The topology of deformation spaces of Kleinian groups. *Ann. of Math. (2)*, 152(3):693–741, 2000.
- [5] P. Appell, É. Goursat, and P. Fatou. *Théorie des Fonctions Algébriques, Tome 2: Fonctions Automorphes*. Gauthier-Villars, Paris, 1930.
- [6] L. Bers. Fiber spaces over Teichmüller spaces. *Acta. Math.*, 130:89–126, 1973.
- [7] L. Bers. Holomorphic families of isomorphisms of Möbius groups. *J. Math. Kyoto Univ.*, 26(1):73–76, 1986.
- [8] F. Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.
- [9] F. Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.
- [10] F. Bonahon. Variations of the boundary geometry of 3-dimensional hyperbolic convex cores. *J. Differential Geom.*, 50(1):1–24, 1998.
- [11] J. Brock and K. Bromberg. On the density of geometrically finite Kleinian groups. *Acta Math.*, 192(1):33–93, 2004.
- [12] J. Brock, K. Bromberg, R. Canary, and Y. Minsky. In preparation.
- [13] K. Bromberg. Projective structures with degenerate holonomy and the Bers density conjecture. *Ann. of Math. (2)*, 166(1):77–93, 2007.
- [14] K. Bromberg and J. Holt. Bumping of exotic projective structures. Preprint.
- [15] K. Bromberg and J. Holt. Self-bumping of deformation spaces of hyperbolic 3-manifolds. *J. Differential Geom.*, 57(1):47–65, 2001.
- [16] R. Canary. Introductory bumponomics: The topology of deformation spaces of hyperbolic 3-manifolds. Preprint, 2007.

- [17] R. Canary, D. Epstein, and P. Green. Notes on notes of Thurston. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pages 3–92. Cambridge Univ. Press, Cambridge, 1987.
- [18] A. Casson and S. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*, volume 9 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988.
- [19] V. Chuckrow. On Schottky groups with applications to Kleinian groups. *Ann. of Math. (2)*, 88:47–61, 1968.
- [20] M. Culler and P. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)*, 117(1):109–146, 1983.
- [21] G. Daskalopoulos, S. Dostoglou, and R. Wentworth. Character varieties and harmonic maps to \mathbf{R} -trees. *Math. Res. Lett.*, 5(4):523–533, 1998.
- [22] G. Daskalopoulos and R. Wentworth. Harmonic maps and Teichmüller theory. In *Handbook of Teichmüller Theory, (A. Papadopoulos, editor), Volume I*, pages 33–110. EMS Publishing House, Zürich, 2007.
- [23] S. Donaldson. Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc. (3)*, 55(1):127–131, 1987.
- [24] D. Dumas. Bear: A tool for studying Bers slices of punctured tori. Free software, available for download from <http://bear.sourceforge.net/>.
- [25] D. Dumas. Grafting, pruning, and the antipodal map on measured laminations. *J. Differential Geometry*, 74:93–118, 2006. Erratum. 77:175–176, 2007.
- [26] D. Dumas. The Schwarzian derivative and measured laminations on Riemann surfaces. *Duke Math. J.*, 140(2):203–243, 2007.
- [27] D. Dumas and M. Wolf. Projective structures, grafting, and measured laminations. *Geometry and Topology*, 12(1):351–386, 2008.
- [28] C. Earle. On variation of projective structures. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 87–99, Princeton, N.J., 1981. Princeton Univ. Press.
- [29] J. Eells and L. Lemaire. A report on harmonic maps. *Bull. London Math. Soc.*, 10(1):1–68, 1978.
- [30] J. Eells and L. Lemaire. Another report on harmonic maps. *Bull. London Math. Soc.*, 20(5):385–524, 1988.
- [31] J. Eells and J. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.
- [32] D. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pages 113–253. Cambridge Univ. Press, Cambridge, 1987.

- [33] G. Faltings. Real projective structures on Riemann surfaces. *Compositio Math.*, 48(2):223–269, 1983.
- [34] A. Fathi, F. Laudenbach, and V. Poenaru. *Travaux de Thurston sur les surfaces*, volume 66 of *Astérisque*. Société Mathématique de France, Paris, 1979. Séminaire Orsay, With an English summary.
- [35] E. Frenkel and D. Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2004.
- [36] D. Gallo, M. Kapovich, and A. Marden. The monodromy groups of Schwarzian equations on closed Riemann surfaces. *Ann. of Math. (2)*, 151(2):625–704, 2000.
- [37] F. Gardiner. *Teichmüller theory and quadratic differentials*. Pure and Applied Mathematics. John Wiley & Sons Inc., New York, 1987.
- [38] W. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.
- [39] W. Goldman. Projective structures with Fuchsian holonomy. *J. Differential Geom.*, 25(3):297–326, 1987.
- [40] W. Goldman. Geometric structures on manifolds and varieties of representations. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 169–198. Amer. Math. Soc., Providence, RI, 1988.
- [41] W. Goldman. Topological components of spaces of representations. *Invent. Math.*, 93(3):557–607, 1988.
- [42] W. Goldman. Ergodic theory on moduli spaces. *Ann. of Math. (2)*, 146(3):475–507, 1997.
- [43] R. Gunning. *Lectures on Riemann surfaces*. Princeton Mathematical Notes. Princeton University Press, Princeton, N.J., 1966.
- [44] R. Gunning. *Lectures on vector bundles over Riemann surfaces*. University of Tokyo Press, Tokyo, 1967.
- [45] R. Gunning. Special coordinate coverings of Riemann surfaces. *Math. Ann.*, 170:67–86, 1967.
- [46] R. Gunning. Affine and projective structures on Riemann surfaces. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 225–244, Princeton, N.J., 1981. Princeton Univ. Press.
- [47] D. Hejhal. Monodromy groups and linearly polymorphic functions. *Acta Math.*, 135(1):1–55, 1975.
- [48] D. Hejhal. Monodromy groups and Poincaré series. *Bull. Amer. Math. Soc.*, 84(3):339–376, 1978.
- [49] M. Heusener and J. Porti. The variety of characters in $\mathrm{PSL}_2(\mathbb{C})$. *Bol. Soc. Mat. Mexicana (3)*, 10(Special Issue):221–237, 2004.

- [50] J. Holt. Some new behaviour in the deformation theory of Kleinian groups. *Comm. Anal. Geom.*, 9(4):757–775, 2001.
- [51] J. Hubbard. The monodromy of projective structures. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 257–275, Princeton, N.J., 1981. Princeton Univ. Press.
- [52] J. Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*. Matrix Editions, Ithaca, NY, 2006.
- [53] J. Hubbard and H. Masur. Quadratic differentials and foliations. *Acta Math.*, 142(3-4):221–274, 1979.
- [54] A. Huber. On subharmonic functions and differential geometry in the large. *Comment. Math. Helv.*, 32:13–72, 1957.
- [55] A. Huber. Zum potentialtheoretischen Aspekt der Alexandrowschen Flächentheorie. *Comment. Math. Helv.*, 34:99–126, 1960.
- [56] Y. Iwayoshi and M. Taniguchi. *An introduction to Teichmüller spaces*. Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.
- [57] K. Ito. Exotic projective structures and quasi-Fuchsian space. *Duke Math. J.*, 105(2):185–209, 2000.
- [58] K. Ito. Schottky groups and Bers boundary of Teichmüller space. *Osaka J. Math.*, 40(3):639–657, 2003.
- [59] K. Ito. Grafting and components of quasi-Fuchsian projective structures. In *Spaces of Kleinian groups*, volume 329 of *London Math. Soc. Lecture Note Ser.*, pages 355–373. Cambridge Univ. Press, Cambridge, 2006.
- [60] K. Ito. Exotic projective structures and quasi-Fuchsian space, II. *Duke Math. J.*, 140(1):85–109, 2007.
- [61] T. Jørgensen. On discrete groups of Möbius transformations. *Amer. J. Math.*, 98(3):739–749, 1976.
- [62] J. Jost. *Compact Riemann surfaces*. Universitext. Springer-Verlag, Berlin, third edition, 2006.
- [63] Y. Kamishima. Conformally flat manifolds whose development maps are not surjective. I. *Trans. Amer. Math. Soc.*, 294(2):607–623, 1986.
- [64] Y. Kamishima and S. Tan. Deformation spaces on geometric structures. In *Aspects of low-dimensional manifolds*, volume 20 of *Adv. Stud. Pure Math.*, pages 263–299. Kinokuniya, Tokyo, 1992.
- [65] M. Kapovich. On monodromy of complex projective structures. *Invent. Math.*, 119(2):243–265, 1995.
- [66] M. Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001.
- [67] F. Klein. *Ausgewählte Kapitel aus der Theorie der linearen Differentialgleichungen zweiter Ordnung*, volume 1. Göttingen, 1891.

- [68] F. Klein. *Vorlesungen Über die Hypergeometrische Funktion*. Springer-Verlag, Berlin, 1933.
- [69] S. Kojima. Circle packing and Teichmüller spaces. To appear in *Handbook of Teichmüller Theory*, (A. Papadopoulos, editor), Volume II, EMS Publishing House, 2009.
- [70] S. Kojima, S. Mizushima, and S. Tan. Circle packings on surfaces with projective structures: a survey. In *Spaces of Kleinian groups*, volume 329 of *London Math. Soc. Lecture Note Ser.*, pages 337–353. Cambridge Univ. Press, Cambridge, 2006.
- [71] N. Korevaar and R. Schoen. Global existence theorems for harmonic maps: finite rank spaces and an approach to rigidity for smooth actions. Preprint.
- [72] N. Korevaar and R. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.
- [73] N. Korevaar and R. Schoen. Global existence theorems for harmonic maps to non-locally compact spaces. *Comm. Anal. Geom.*, 5(2):333–387, 1997.
- [74] I. Kra. Deformations of Fuchsian groups. *Duke Math. J.*, 36:537–546, 1969.
- [75] I. Kra. Deformations of Fuchsian groups. II. *Duke Math. J.*, 38:499–508, 1971.
- [76] I. Kra. A generalization of a theorem of Poincaré. *Proc. Amer. Math. Soc.*, 27:299–302, 1971.
- [77] I. Kra and B. Maskit. Remarks on projective structures. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 343–359, Princeton, N.J., 1981. Princeton Univ. Press.
- [78] R. Kulkarni and U. Pinkall. A canonical metric for Möbius structures and its applications. *Math. Z.*, 216(1):89–129, 1994.
- [79] O. Lehto. *Univalent functions and Teichmüller spaces*, volume 109 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1987.
- [80] G. Levitt. Foliations and laminations on hyperbolic surfaces. *Topology*, 22(2):119–135, 1983.
- [81] F. Luo. Monodromy groups of projective structures on punctured surfaces. *Invent. Math.*, 111(3):541–555, 1993.
- [82] A. Marden. *Outer Circles: An Introduction to Hyperbolic 3-Manifolds*. Cambridge University Press, Cambridge, 2007.
- [83] B. Maskit. On a class of Kleinian groups. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 442:8, 1969.
- [84] K. Matsuzaki. The interior of discrete projective structures in the Bers fiber. *Ann. Acad. Sci. Fenn. Math.*, 32(1):3–12, 2007.
- [85] K. Matsuzaki and M. Taniguchi. *Hyperbolic manifolds and Kleinian groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1998. , Oxford Science Publications.

- [86] C. McMullen. Complex earthquakes and Teichmüller theory. *J. Amer. Math. Soc.*, 11(2):283–320, 1998.
- [87] C. Mese. Harmonic maps between surfaces and Teichmüller spaces. *Amer. J. Math.*, 124(3):451–481, 2002.
- [88] C. Mese. Harmonic maps into spaces with an upper curvature bound in the sense of Alexandrov. *Math. Z.*, 242(4):633–661, 2002.
- [89] C. Mese. Uniqueness theorems for harmonic maps into metric spaces. *Commun. Contemp. Math.*, 4(4):725–750, 2002.
- [90] J. Morgan and P. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.
- [91] Z. Nehari. The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.*, 55:545–551, 1949.
- [92] K. Ohshika. Divergence, exotic convergence, and self-bumping in quasi-Fuchsian spaces. In preparation.
- [93] B. Osgood and D. Stowe. The Schwarzian derivative and conformal mapping of Riemannian manifolds. *Duke Math. J.*, 67(1):57–99, 1992.
- [94] R. Penner and J. Harer. *Combinatorics of train tracks*, volume 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992.
- [95] H. Poincaré. Sur les groupes des équations linéaires. *Acta Math.*, 4(1):201–312, 1884.
- [96] C. Pommerenke. *Univalent functions*. Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, *Studia Mathematica/Mathematische Lehrbücher*, Band XXV.
- [97] A. Rapinchuk, V. Benyash-Krivetz, and V. Chernousov. Representation varieties of the fundamental groups of compact orientable surfaces. *Israel J. Math.*, 93:29–71, 1996.
- [98] Y. Reshetnyak. Two-dimensional manifolds of bounded curvature. In *Geometry. IV: Nonregular Riemannian geometry*, Encyclopaedia of Mathematical Sciences, pages 3–163. Springer-Verlag, Berlin, 1993. Translation of *Geometry, 4 (Russian)*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989.
- [99] Y. Reshetnyak. On the conformal representation of Alexandrov surfaces. In *Papers on analysis*, volume 83 of *Rep. Univ. Jyväskylä Dep. Math. Stat.*, pages 287–304. Univ. Jyväskylä, Jyväskylä, 2001.
- [100] B. Riemann. Vorlesungen über die hypergeometrische Reihe. In *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachträge*, pages 667–692. Springer-Verlag, Berlin, 1990.
- [101] K. Scannell. *Flat conformal structures and causality in de Sitter manifolds*. PhD thesis, University of California, Los Angeles, 1996.
- [102] K. Scannell. Flat conformal structures and the classification of de Sitter manifolds. *Comm. Anal. Geom.*, 7(2):325–345, 1999.

- [103] K. Scannell and M. Wolf. The grafting map of Teichmüller space. *J. Amer. Math. Soc.*, 15(4):893–927 (electronic), 2002.
- [104] R. Schoen. Analytic aspects of the harmonic map problem. In *Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983)*, volume 2 of *Math. Sci. Res. Inst. Publ.*, pages 321–358. Springer, New York, 1984.
- [105] R. Sharpe. *Differential geometry*, volume 166 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Cartan’s generalization of Klein’s Erlangen program, With a foreword by S. S. Chern.
- [106] H. Shiga. Projective structures on Riemann surfaces and Kleinian groups. *J. Math. Kyoto Univ.*, 27(3):433–438, 1987.
- [107] H. Shiga and H. Tanigawa. Projective structures with discrete holonomy representations. *Trans. Amer. Math. Soc.*, 351(2):813–823, 1999.
- [108] D. Sullivan. Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups. *Acta Math.*, 155(3-4):243–260, 1985.
- [109] D. Sullivan and W. Thurston. Manifolds with canonical coordinate charts: some examples. *Enseign. Math. (2)*, 29(1-2):15–25, 1983.
- [110] H. Tanigawa. Grafting, harmonic maps and projective structures on surfaces. *J. Differential Geom.*, 47(3):399–419, 1997.
- [111] H. Tanigawa. Divergence of projective structures and lengths of measured laminations. *Duke Math. J.*, 98(2):209–215, 1999.
- [112] W. Thurston. Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle. Preprint.
- [113] W. Thurston. Geometry and topology of three-manifolds. Princeton lecture notes, 1979.
- [114] W. Thurston. Earthquakes in two-dimensional hyperbolic geometry. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 91–112. Cambridge Univ. Press, Cambridge, 1986.
- [115] W. Thurston. Minimal stretch maps between hyperbolic surfaces. Unpublished preprint, 1986.
- [116] W. Thurston. Zippers and univalent functions. In *The Bieberbach conjecture (West Lafayette, Ind., 1985)*, volume 21 of *Math. Surveys Monogr.*, pages 185–197. Amer. Math. Soc., Providence, RI, 1986.
- [117] W. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.
- [118] M. Wolf. The Teichmüller theory of harmonic maps. *J. Differential Geom.*, 29(2):449–479, 1989.
- [119] M. Wolf. On realizing measured foliations via quadratic differentials of harmonic maps to \mathbf{R} -trees. *J. Anal. Math.*, 68:107–120, 1996.

- [120] D. Wright. The shape of the boundary of the Teichmüller space of once-punctured tori in Maskit's embedding. Unpublished preprint, 1987.